" IN THE NAME OF ALLAH, MOST GRACIOUS, MOST MERCIFUL"

Topics in Seminear-ring Theory

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To

My Mother

Whose prayers have always been a source of great inspiration to me

and

My Wife

For her patience and understanding of the inevitable neglect of my obligation during the course of my research work.

Abstract

The idea of a seminear-ring was introduced in [9], as an algebraic system that can be constructed from a set S with two binary operations : addition $+$ and multiplication ., such that $(S,+)$ and $(S,.)$ are semigroups and one distributive law is satisfied. A seminear-ring S is called distributively generated (d.g.) if S contains a multiplicative subsemigroup $(T,.)$ of distributive elements which generates $(S,+)$. Unlike the near-rings case for which a rich theory has already been developed, very little seems to be known about seminear-rings. The aim of this dissertation consists mainly of two goals. The first is to generalize some results which are known in the theory of near-rings. The second goal of this thesis appears mainly in the last 6 chapters in which we obtain some results about seminear-rings of endomorphisms.

In chapter 1, the definitions and basic concepts about seminear-rings are given; e.g. an arbitrary seminear-ring can be embedded in a seminear-ring of the form $M(S)$.

Fröhlich [1], [2] and Meldrum [6] have given some results concerning free d.g. near-rings in a variety $\mathcal{V}$. In chapter 2, we generalize some of these results to free d.g. seminear-rings and we can prove the existence of free (S,T)-semigroups on a set X in a variety $\mathcal{V}$. In section 2.4, we prove a theorem which asserts that not every d.g. seminear-ring has a faithful representation. This would generalize the result which was given by Meldrum [6] for the near-ring case.

Chapter 3 gives an overview of strong semilattices of near-rings and of rings. In this context we show that a strong semilattice of near-rings is a seminear-ring while a strong semilattice of rings is a semiring.

Chapter 4 is designed to be a preparatory chapter for the remaining part of the thesis. It explains the main plan which will be followed in all the last 6 chapters. It also includes some basic ideas and results which are of great use in the remaining work.

Throughout chapter 5 to chapter 10 we will be considering seminear-rings of endomorphisms of some special types. In each chapter we consider some groups G_α, where each α belongs to the semilattice Y, then we study the structure of the corresponding strong semilattice S of groups $G_\alpha, \alpha \in Y$. For each group

G_α there will be a d.g. near-ring $E(G_\alpha)$ generated by $\mathrm{End}(G_\alpha)$, the set of all endomorphisms of G_α. On the other hand, considering S, the semilattice of the groups G_α, then End(S), the set of all endomorphisms of S, will generate a d.g. seminear-ring $E(S)$. So we study the structure of $E(S)$ with its connection to $\{E(G_\alpha); \alpha \in Y\}$ which asserts that $E(S)$, indeed, forms a Clifford semigroup.

For the background in semigroup theory we refer to Howie [4] which is the standard book in that subject.

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Declaration

I declare that this thesis was composed by myself and that the work contained therein is my own, except where explicitly stated otherwise in the text.

(*Mohammad Samman*)

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Chapter 1

Basic concepts

Since semigroups are widely involved in the theory of seminear-rings, we start this chapter by giving some background in semigroup theory which is needed at several stages throughout this work.

1.1 Background in semigroup theory

Let S be a semigroup. If S has no identity element, then we can adjoin an identity element 1 to the set S by defining $1.s = s.1 = s$, for all $s \in S$, and 1.1=1. Then $S \bigcup \{1\}$ becomes a semigroup with an identity element 1 (usually denoted by S^1). A subsemigroup H of a semigroup S is called a subgroup of S if H itself is a group. A semigroup S is called a union of groups if each element of S is contained in some subgroup of S. Thus if s is an element of such a semigroup S, then $s \in G$, where G is a subgroup of S. Let T be a subset of a semigroup S. The intersection of all subsemigroups of S containing T is a subsemigroup of S denoted by sg$< T >$, and called the subsemigroup of S generated by T. Also sg$< T >$ can be described as the set of all elements of S which are expressible as finite products of elements of T. If sg$< T > = S$ then T is called a set of generators of S or a generating set of S. Let S and H be two semigroups. A function $\theta : S \longrightarrow H$ is called a semigroup homomorphism if $(ab)\theta = (a)\theta(b)\theta$ for all $a, b \in S$. Semigroup monomorphisms, epimorphisms and isomorphisms are defined as usual. Let θ be a homomorphism of a semigroup S into a semigroup

H. The kernel of θ is defined by $\ker\theta := \{(x,y) \in S \times S; (x)\theta = (y)\theta\}$. It is easily seen that $\ker\theta$ is a congruence on S. In general, if ρ is a congruence (kernel) on S, then S/ρ denotes the set of all equivalence classes of S determined by ρ. Thus $S/\rho := \{a\rho; a \in S\}$. The set S/ρ can be made into a semigroup by defining $(a\rho)(b\rho) = (ab)\rho$. We refer to S/ρ as the factor semigroup of S modulo ρ. A (natural) mapping $\eta : S \longrightarrow S/\rho$ defined by $(a)\eta = a\rho$ is a semigroup epimorphism.

1.2 Seminear-rings : basic concepts and examples

Definition 1.2.1 A set S with two binary operations $+$ and . is called a seminear-ring if it satisfies :

(i) $(S,+)$ and $(S,.)$ are semigroups, and

(ii) one distributive law is satisfied, namely,

$a(b+c) = ab + ac$ for all $a, b, c \in S$.

S as defined above is called a left seminear-ring (which we are considering throughout this thesis) and it is called a right seminear-ring if it satisfies (i) and

(ii)$'$ $(a+b)c = ac + bc$, for all $a, b, c \in S$.

We note that if all the conditions (i), (ii) and (ii)$'$ are satisfied such that $(S,+)$ is abelian then S is called a semiring.

The following are two examples of seminear-rings.

Example 1.2.2 Let N^+ be the set of all non-negative integers with the usual addition and multiplication. Then $(N^+,+,.)$ is a seminear-ring. Indeed, it is a semiring.

Example 1.2.3 Let S be an additive semigroup. Let $M(S) := \{\alpha : S \longrightarrow S\}$ be the set of all mappings of S into itself, with the operations of pointwise addition:

$$s(\alpha+\beta) = s\alpha + s\beta \text{ for all } s \in S,$$

and multiplication given by composition of maps :

$$s(\alpha\beta) = (s\alpha)\beta \text{ for all } s \in S.$$

Then $(M(S), +, .)$ is a seminear-ring.

Definition 1.2.4 Let $(S, +, .)$ be a seminear-ring. Then a non-empty subset T of S is called a subseminear-ring if $(T, +, .)$ is a seminear-ring with the operations induced by the operations in S.

The following easy lemma gives a criterion for a set to be a subseminear-ring.

Lemma 1.2.5 Let T be a non-empty subset of the seminear-ring S. Then T is a subseminear-ring of S if and only if T is closed under addition and multiplication.

Definition 1.2.6 Let $(S, +, .)$ and $(S', +, .)$ be two seminear-rings. A mapping $\phi : S \longrightarrow S'$ is called a seminear-ring homomorphism if

$$\begin{aligned} (a + b)\phi &= (a)\phi + (b)\phi \\ (ab)\phi &= (a)\phi(b)\phi \text{ for all } a, b \in S. \end{aligned}$$

Usually in our context when clearly dealing with a seminear-ring homomorphism, we write only a homomorphism. The concepts of monomorphism, epimorphism, isomorphism and endomorphism are defined in the usual manner.

Unlike the rings or near-rings case, we define the kernel of a homomorphism as follows.

Definition 1.2.7 Let ϕ be a homomorphism of a seminear-ring S into a seminear-ring S'. We define the kernel of ϕ by:

$$\ker \phi := \{(x, y) \in S \times S; (x)\phi = (y)\phi\}.$$

It is easy to see that $\ker \phi$ is a congruence on S. We can see that in the near-ring case, this definition reduces to the usual one of an ideal of S.

Definition 1.2.8 Let k be a seminear-ring congruence (kernel) on a seminear-ring S. We define the quotient of S by k by $S/k := \{ak; a \in S\}$. Then $(S/k, +, .)$ is a seminear-ring which is called the quotient seminear-ring of S by k, where the operations $+$ and $.$ are defined as : $ak + bk = (a + b)k$ and $(ak)(bk) = (ab)k$.

The above operations are well-defined, since for $a, a', b, b' \in S$, if we suppose that $ak = a'k$ and $bk = b'k$ then $(a, a') \in k$ and $(b, b') \in k$. Thus $(a + b, a' + b') \in k$, which gives $(a + b)k = (a' + b')k$. Similarly for multiplication we have $(a, a') \in k$ and $(b, b') \in k$ imply that $(ab, a'b') \in k$, giving that $(ab)k = (a'b')k$. Hence the given operations are well-defined.

Definition 1.2.9 Let S be a seminear-ring with a congruence (kernel) k. We define the natural mapping $\eta : S \longrightarrow S/k$ by $(a)\eta := ak$. Then η is a homomorphism of S onto S/k.

Homomorphism theorems can be stated analogously to the case of near-rings. So we can state the following.

Theorem 1.2.10 Let ϕ be a homomorphism from a seminear-ring S to a seminear-ring S' with kernel k. Then

$$(S)\phi \cong S/k.$$

If ϕ is a homomorphism from S to a seminear-ring H such that $k \subseteq \ker \phi$, then there is a unique homomorphism

$$\psi : S/k \longrightarrow H$$

such that $\eta\psi = \phi$, where η is the natural homomorphism associated with k, and ψ is given by $(ak)\psi = (a)\phi$.

Theorem 1.2.11 Let S be a semigroup. Let T be a subsemigroup of S and ρ a congruence on S. Then $T\rho$,(the set $\{a\rho; a \in T\}$ of all equivalence classes of a under ρ), is a subsemigroup of S , $\rho \cap T^2$ is a congruence on T and

$$T/\rho \cap T^2 \cong T\rho/\rho,$$

where $T\rho/\rho$ is the semigroup of equivalence classes under ρ contained in $T\rho$.

1.3 An embedding result

Considering the seminear-ring given in example 1.2.3, we can prove an embedding result concerning seminear-rings.

Theorem 1.3.1 Every seminear-ring can be embedded in a seminear-ring of the form $M(S)$.

Proof Let $(S, +, .)$ be a seminear-ring. If S does not contain a multiplicative identity, we adjoin 1 to it and write $S^1 = (S, +) \bigcup \{1\}$. Define addition in S^1 by putting $s + 1 = 1 + s = s$ for all $s \in S^1$.

For each $s \in S$, define a map ρ_s where

$$\rho_s : S^1 \longrightarrow S^1 \text{ is given by}$$

$$(a)\rho_s = as \text{ for all } a \in S, \text{ and}$$

$$(1)\rho_s = s.$$

Then $\rho_s \in M(S^1)$.

Consider the map $\rho : S \longrightarrow M(S^1)$ given by

$$(s)\rho = \rho_s \ .$$

We show that ρ is a homomorphism.

For $s_1, s_2 \in S$ we have

$$\begin{aligned}(1)(s_1 + s_2)\rho = (1)\rho_{s_1+s_2} = 1(s_1 + s_2) = s_1 + s_2 &= (1)\rho_{s_1} + (1)\rho_{s_2} \\ &= (1)(\rho_{s_1} + \rho_{s_2}) \\ &= (1)[(s_1)\rho + (s_2)\rho],\end{aligned}$$

and for all $x \in S$, we have

$$\begin{aligned}(x)(s_1 + s_2)\rho = (x)\rho_{s_1+s_2} = x(s_1 + s_2) = xs_1 + xs_2 &= (x)\rho_{s_1} + (x)\rho_{s_2} \\ &= (x)(\rho_{s_1} + \rho_{s_2}) \\ &= (x)[(s_1)\rho + (s_2)\rho].\end{aligned}$$

This shows that

$$(s_1 + s_2)\rho = (s_1)\rho + (s_2)\rho \ .$$

Also we have

$(1)(s_1s_2)\rho = (1)\rho_{s_1s_2} = s_1s_2$ and $(1)\rho_{s_1}\rho_{s_2} = ((1)\rho_{s_1})\rho_{s_2} = (s_1)\rho_{s_2} = s_1s_2$,

and for all $x \in S$, we have

$(x)(s_1s_2)\rho = (x)\rho_{s_1s_2} = x(s_1s_2) = (xs_1)s_2$, and

$(x)(s_1\rho)(s_2\rho) = (x)(\rho_{s_1})(\rho_{s_2}) = (x\rho_{s_1})\rho_{s_2} = (xs_1)\rho_{s_2} = (xs_1)s_2$

which implies that

$$(s_1 s_2)\rho = (s_1)\rho(s_2)\rho.$$

Thus ρ is a homomorphism.

Moreover, ρ is a monomorphism since if $s_1 \neq s_2$ in S, then $(1)\rho_{s_1} \neq (1)\rho_{s_2}$ and so $\rho_{s_1} \neq \rho_{s_2}$ or $(s_1)\rho \neq (s_2)\rho$ showing that ρ is a monomorphism.

Hence ρ is an embedding of S in $M(S^1)$.

Chapter 2

Free d.g. seminear-rings

Our purpose in this chapter is to study the concept of free d.g. seminear-rings and develop some related results; in addition we study the representation of d.g. seminear-rings. We start by considering a special class of seminear-rings which has a good behaviour with respect to distributivity, that is the class of d.g. seminear-rings. Then we proceed to develop and study free d.g. seminear-rings. This will lead us to generalize some results which are already known in the theory of near-rings. In this context we have to start by giving the basic ideas about d.g. seminear-rings in the following section.

2.1 Distributively generated seminear-rings

Definition 2.1.1 Let S be a seminear-ring. An element $d \in S$ is called distributive if

$$(a+b)d = ad + bd \qquad \text{for all } a, b \in S \ .$$

Example 2.1.2 Consider $M(S)$, the seminear-ring given in example 1.2.3. Let $\mathrm{End}(S)$ be the set of all endomorphisms of S. Then the elements of $\mathrm{End}(S)$ are distributive.

Lemma 2.1.3 Let S be a seminear-ring. The set of all distributive elements of S forms a semigroup under multiplication.

Definition 2.1.4 A seminear-ring S is called distributively generated (denoted by d.g. seminear-ring) if S contains a multiplicative subsemigroup $(T, .)$ of distributive elements which generates $(S,+)$. i.e. $(S,+) = \text{sg}< T >$. We usually denote such a d.g. seminear-ring by (S,T).

Remark 2.1.5 We remark that the set T of distributive elements which generate S as an additive semigroup need not be the set of all distributive elements of S.

Lemma 2.1.6 Let S be a seminear-ring. If T is a multiplicatively closed subset consisting of distributive elements of S, then T generates a d.g. seminear-ring M such that $M = \text{sg}< T, + >$.

Proof We need to show that $\text{sg}< T, + >$ is closed under multiplication. Let $x, y \in \text{sg}< T >$, then
$x = x_1 + x_2 + \cdots + x_n$, $y = y_1 + y_2 + \cdots + y_m$, where $x_i, y_j \in T$, and
$1 \leq i \leq n$, $1 \leq j \leq m$. So we have

$$\begin{aligned} xy &= (x_1 + x_2 + \cdots + x_n)(y_1 + y_2 + \cdots + y_m) \\ &= (x_1 + x_2 + \cdots + x_n)y_1 + \cdots + (x_1 + x_2 + \cdots + x_n)y_m \\ &= x_1y_1 + \cdots + x_ny_1 + \cdots + x_1y_m + \cdots + x_ny_m. \end{aligned}$$

Since $x_iy_j \in T$, for all $1 \leq i \leq n$, $1 \leq j \leq m$, $xy \in \text{sg}< T, + >$, which completes the proof.

A natural example of d.g. seminear-rings which we will be considering in the last 6 chapters of this thesis is the following.

Example 2.1.7 Let S be a semigroup. Then from example 2.1.2 and lemma 2.1.3, End(S), the set of all endomorphisms of S is a semigroup of distributive elements of the seminear-ring M(S). By lemma 2.1.6, End(S) will generate a d.g. seminear-ring which we usually denote by ($E(S)$,End(S)) or simply $E(S)$.

2.2 Further definitions and required results

In the remaining part of this chapter we will be considering varieties of semigroups. By a variety of semigroups we mean the class of all semigroups satisfying a given set of laws. For example, the variety of commutative semigroups is the class of all semigroups which satisfy the law $x+y=y+x$. Varieties of d.g. seminear-rings can be defined in the same way, but by using laws involving both addition and multiplication. The only varieties of d.g. seminear-rings we will be considering will be those satisfying additive laws.

Definition 2.2.1 Let T be a multiplicative semigroup. A semigroup H is called a T-semigroup if there exists a homomorphism $\theta : T \longrightarrow \mathrm{End}(H)$. We write ht for $h(t\theta)$.

If S is a seminear-ring, then a semigroup H is called an S-module if there is a seminear-ring homomorphism $\theta : S \longrightarrow M(H)$.

Such a homomorphism is called a representation of S. A representation θ is called faithful if ker θ is trivial.

Definition 2.2.2 Let S be a seminear-ring, and let T be a multiplicative subsemigroup of S. Let H and K be two S-modules. Then a homomorphism $\theta : H \longrightarrow K$ is called a T-homomorphism if $(ht)\theta = (h)\theta t$ for all $h \in H$, $t \in T$.

The following three lemmas are generalizations of results given, for the d.g. near-ring case, by Meldrum [7].

Lemma 2.2.3 Let (S,T) be a d.g. seminear-ring, and let H and K be two S-modules. Then a homomorphism $\theta : H \longrightarrow K$ is an S-homomorphism if and only if it is a T-homomorphism.

Proof If $\theta : H \longrightarrow K$ is an S-homomorphism then obviously it is a T-homomorphism. Conversely, let θ be a T-homomorphism then

$$(ht)\theta = (h\theta)t \ \text{ for all}\, h \in H,\, t \in T.$$

A typical element $s \in S$, can be expressed in the form

$$s = t_1 + \ldots + t_n,\ t_i \in T,\ 1 \leq i \leq n.$$

Thus

$$(hs)\theta = (h(t_1 + \ldots + t_n))\theta$$

$$
\begin{aligned}
&= (ht_1 + \ldots + ht_n)\theta \\
&= (ht_1)\theta + \ldots + (ht_n)\theta \\
&= (h\theta)t_1 + \ldots + (h\theta)t_n \\
&= (h\theta)(t_1 + \ldots + t_n) \\
&= (h\theta)s.
\end{aligned}
$$

Hence θ is an S-homomorphism.

Definition 2.2.4 Let (N, T) and (M, U) be two d.g. seminear-rings. A seminear-ring homomorphism $\theta : N \longrightarrow M$ such that $T\theta \leq U$ is called a d.g. homomorphism.

Now we state a result concerning d.g. homomorphisms.

Lemma 2.2.5 Let (N, T) and (M, U) be two d.g. seminear-rings. If ϕ is a map from N to M which is both a semigroup homomorphism from $(N, +)$ to $(M, +)$ and a semigroup homomorphism from $(T, .)$ to $(U, .)$, then it is a d.g. seminear-ring homomorphism from (N, T) to (M, U).

Proof We only need to show that ϕ is a semigroup homomorphism from $(N, .)$ to $(M, .)$. Let $x, y \in N$, then $x = x_1 + x_2 + \cdots + x_q$, $y = y_1 + y_2 + \cdots + y_l$, where $x_i, y_j \in T$, and $1 \leq i \leq q$, $1 \leq j \leq l$. So

$$
\begin{aligned}
xy &= (x_1 + \cdots + x_q)(y_1 + \cdots + y_l) \\
&= (x_1 + \cdots + x_q)y_1 + \cdots + (x_1 + \cdots + x_q)y_l \\
&= (x_1y_1 + \cdots + x_qy_1) + \cdots + (x_1y_l + \cdots + x_qy_l).
\end{aligned}
$$

Thus we have

$$
\begin{aligned}
(xy)\phi &= ((x_1y_1 + \cdots + x_qy_1) + \cdots + (x_1y_l + \cdots + x_qy_l))\phi \\
&= (x_1y_1)\phi + \cdots + (x_qy_1)\phi + \cdots + (x_1y_l)\phi + \cdots + (x_qy_l)\phi \\
&= (x_1)\phi(y_1)\phi + \cdots + (x_q)\phi(y_1)\phi + \cdots + (x_1)\phi(y_l)\phi + \cdots + (x_q)\phi(y_l)\phi \\
&= [(x_1)\phi + \cdots + (x_q)\phi](y_1)\phi + \cdots + [(x_1)\phi + \cdots + (x_q)\phi](y_l)\phi \\
&= [(x_1)\phi + \cdots + (x_q)\phi][(y_1)\phi + \cdots + (y_l)\phi]
\end{aligned}
$$

$$
\begin{aligned}
&= (x_1 + \cdots + x_q)\phi(y_1 + \cdots + y_l)\phi \\
&= (x)\phi(y)\phi.
\end{aligned}
$$

Hence ϕ is a semigroup homomorphism from $(N, .)$ to $(M, .)$ which completes the proof.

Now for a given variety of semigroups we may specify a variety of seminear-rings as the following definition tells us.

Definition 2.2.6 Let $\mathcal{V}$ be a variety of semigroups. If the additive semigroup of the seminear-ring S lies in $\mathcal{V}$, then we say that the seminear-ring S belongs to $\mathcal{V}$.

Lemma 2.2.7 Let $w(x_1, \ldots, x_n)$ be a semigroup word in n variables $x_1, \ldots, x_n$. Then $hw(s_1, \ldots, s_n) = w(hs_1, \ldots, hs_n)$, whenever $s_1, \ldots, s_n$ lie in a seminear-ring S and h lies in an S-module H.

Proof Let $w(x_1, \ldots, x_n)$ be a word in the variables $x_1, \ldots, x_n$. Then w can be expressed as $w(s_1, \ldots, s_n) = s_{i_1} + \ldots + s_{i_m}$, where $s_{i_j} \in \{s_1, \ldots, s_n\}$ for $1 \leq j \leq m$. So for $h \in H$, we have

$$
\begin{aligned}
hw(s_1, \ldots, s_n) &= h(s_{i_1} + \cdots + s_{i_m}) \\
&= (hs_{i_1} + \ldots + hs_{i_m}) \\
&= w(hs_1, \ldots, hs_n).
\end{aligned}
$$

Using lemma 2.2.7 we can prove the following theorem.

Theorem 2.2.8 Let S be a seminear-ring with a faithful representation on the S-module H. If H belongs to variety $\mathcal{V}$, then so does S.

Proof. Let $w(x_1, \ldots, x_n) = u(x_1, \ldots, x_n)$ be a law in the variety $\mathcal{V}$ for which $H \in \mathcal{V}$. By lemma 2.2.7, for $s_1, \ldots, s_n \in S$ and $h \in H$, we have

$$
\begin{aligned}
hw(s_1, \ldots, s_n) &= w(hs_1, \ldots, hs_n) \text{ , and} && (2.1) \\
hu(s_1, \ldots, s_n) &= u(hs_1, \ldots, hs_n) \text{ for all } h \in H. && (2.2)
\end{aligned}
$$

Since the law $w = u$ holds in $H \in \mathcal{V}$ and $hs_i \in H$, it follows that

$$w(hs_1, ..., hs_n) = u(hs_1, ..., hs_n).$$

Substituting from (2.1) and (2.2) yields

$$hw(s_1, ..., s_n) = hu(s_1, ..., s_n) \text{ for all } h \in H,$$

so $w(s_1, ..., s_n)$ and $u(s_1, ..., s_n)$ induce the same map on H. As H is faithful, we conclude that

$$w(s_1, ..., s_n) = u(s_1, ...s_n)$$

which means that the law $w = u$ holds in $(S, +)$. But this is true for all the laws of $\mathcal{V}$. Hence $(S, +) \in \mathcal{V}$.

Definition 2.2.9 Let (S, T) be a d.g. seminear-ring. A representation θ of S is a d.g. representation if $\theta : S \longrightarrow M(H)$ satisfies $T\theta \subseteq \text{End}(H)$, where H is the S-module associated with the representation θ.

We note that a d.g. representation of (S, T) on the S-module H is a d.g. homomorphism from (S, T) to $(E(H),\text{End}(H))$.

Example 2.2.10 Consider the seminear-ring of positive integers Z^+ as a d.g. seminear-ring $(Z^+, 1)$. Then $(Z^+, 1)$ has a d.g. representation on every semigroup. If $\theta : (Z^+, 1) \longrightarrow (E(H),\text{End}(H))$ is a d.g. homomorphism, then Ker $\theta = \{(i, i + n)\}$, where n is the period of H (see Howie [4], page 8).

The above definition leads to the following :

Definition 2.2.11 Let (S, T) be a d.g. seminear-ring. A semigroup H is called an (S, T)-semigroup if (S, T) has a d.g. representation on H.

2.3 Free d.g. seminear-rings

Let $\mathcal{V}$ be a variety of semigroups. Given a set X, $F_{\mathcal{V}}(X)$ denotes the free additive semigroup in $\mathcal{V}$ on X. Let T be a multiplicative semigroup and define the semigroup $\text{Frs}_{\mathcal{V}}(X, T)$ as the free semigroup in the variety $\mathcal{V}$ on the set of symbols:

$$\{x, t_x : x \in X, t \in T\} = T_X.$$

The set T_X is consisting of all elements of X with copies of T. Thus we may write

$$\mathrm{Frs}_{\mathcal{V}}(X,T) = \mathrm{Frs}_{\mathcal{V}} < T_X >.$$

For each $t \in T$, define a map t^* from T_X into $\mathrm{Frs}_{\mathcal{V}}(X,T)$ by

$$\begin{aligned} x.t^* &:= t_x, \\ (m_x)t^* &:= (mt)_x \quad \text{for all } \ x \in X, m \in T, \end{aligned} \tag{2.3}$$

which we extend to be an endomorphism of $\mathrm{Frs}_{\mathcal{V}}(X,T)$. Thus the elements of T are mapped to $\mathrm{End}(\mathrm{Frs}_{\mathcal{V}}(X,T))$, the semigroup of all endomorphisms of $\mathrm{Frs}_{\mathcal{V}}(X,T)$. Indeed this map, $t \longrightarrow t^*$, is a monomorphism of semigroups as the following lemma shows.

Lemma 2.3.1 Let $\lambda : T \longrightarrow \mathrm{End}(\mathrm{Frs}_{\mathcal{V}}(X,T))$ be the map given by $\lambda : t \longrightarrow t^*$ where t^* is defined as above. Then λ is a semigroup monomorphism.

Proof Let $t_1, t_2 \in T$. Then for any $x \in X$ we have

$x(t_1t_2)\lambda = x(t_1t_2)^* = (t_1t_2)_x = (t_{1_x})t_2^* = (x)t_1^*t_2^* = x(t_1\lambda)(t_2\lambda)$,

and for any m_x, where $m \in T$, we have

$m_x(t_1t_2)\lambda = m_x(t_1t_2)^* = (mt_1t_2)_x = (mt_1)_x t_2^* = (m_x)t_1^*t_2^* = m_x(t_1\lambda)(t_2\lambda)$.

Thus $(t_1t_2)\lambda = (t_1\lambda)(t_2\lambda)$ on the set T_X and so on the whole of $\mathrm{Frs}_{\mathcal{V}}(X,T)$ since T_X is a generating set of $\mathrm{Frs}_{\mathcal{V}}(X,T)$. Moreover, λ is injective since if $t_1 \neq t_2$ in T, then by the definition of λ it is clear that $t_1\lambda \neq t_2\lambda$ and hence λ is a monomorphism of semigroups.

Note that we may replace t^* by t itself where no confusion would result; then we can write (2.3) as

$$\begin{aligned} x.t &= t_x, \\ (m_x)t &= (mt)_x \quad \text{for all } \ x \in X, \, m \in T. \end{aligned} \tag{2.4}$$

Now, assume that T is a semigroup of endomorphisms of $\mathrm{Frs}_{\mathcal{V}}(X,T)$ then T generates a d.g. seminear-ring which we will denote $(\mathrm{Frs}_{\mathcal{V}}(T), T)$ and call the free d.g. seminear-ring on T in $\mathcal{V}$. One of our main purposes is to clarify the reason for the name given to $(\mathrm{Frs}_{\mathcal{V}}(T), T)$ as defined above. Before doing so, we give the following lemma which is a consequence of theorem 2.2.8.

Lemma 2.3.2 The free d.g. seminear-ring on T in $\mathcal{V}$ lies in $\mathcal{V}$.

Now we can prove an important result concerning $(\mathrm{Frs}_{\mathcal{V}}(T), T)$. Indeed, it is a generalization of a well-known result for the near-ring case which was given by Meldrum [6].

Theorem 2.3.3 Let $(\mathrm{Frs}_{\mathcal{V}}(T), T)$ be the free d.g. seminear-ring on T in the variety $\mathcal{V}$. Then the following hold :

(i) $(\mathrm{Frs}_{\mathcal{V}}(T), +)$ is the free semigroup in $\mathcal{V}$ on the set T.

(ii) Every T-semigroup H in $\mathcal{V}$ is a $(\mathrm{Frs}_{\mathcal{V}}(T), T)$-semigroup.

(iii) Let (S, U) be a d.g. seminear-ring in $\mathcal{V}$ generated by the multiplicative semigroup U. Then every semigroup homomorhism $\theta : T \longrightarrow U$ can be extended to a d.g. seminear-ring homomorphism from $(\mathrm{Frs}_{\mathcal{V}}(T), T)$ to (S, U).

Proof

(i) Evidently $(\mathrm{Frs}_{\mathcal{V}}(T), +)$ lies in $\mathcal{V}$, by lemma 2.3.2. Let $t_1 + \cdots + t_n$ be an element of $(\mathrm{Frs}_{\mathcal{V}}(T), +)$, where $t_i \in T, 1 \leq i \leq n$. Consider an element $x \in X$, then by (2.4) we have

$$x(t_1 + \cdots + t_n) = t_{1x} + \cdots + t_{nx}.$$

Consider sg$< t_x; t \in T >$, the semigroup which is generated by the set $\{t_x; t \in T\}$. For each $x \in X$, define a map θ_x where

$$\theta_x : (\mathrm{Frs}_{\mathcal{V}}(T), +) \longrightarrow \mathrm{sg}< t_x; t \in T > \text{ given by}$$

$$(t_1 + \cdots + t_n)\theta_x = x(t_1 + \cdots + t_n) = t_{1x} + \cdots + t_{nx} .$$

The map θ_x is well defined because if we assume that $a = b$ in $(\mathrm{Frs}_{\mathcal{V}}(T), +)$ where $a = t_1 + \cdots + t_n$, $b = k_1 + \cdots + k_m$, $t_i, k_j \in T$, $1 \leq i \leq n$, $1 \leq j \leq m$, then

$$x(t_1+\cdots+t_n)=x(k_1+\cdots+k_m)\ ,$$

$$(t_1+\cdots+t_n)\theta_x=(k_1+\cdots+k_m)\theta_x.$$

It follows that

$$(a)\theta_x=(b)\theta_x.$$

That is θ_x is well defined.

Now we show that $(\mathrm{Frs}_{\mathcal{V}}(T),+)\cong \mathrm{sg}< t_x;t\in T>$.

Let $a,b\in(\mathrm{Frs}_{\mathcal{V}}(T),+)$. Then $a=t_1+\cdots+t_n$, $b=k_1+\cdots+k_m$, where $t_i,k_j\in T, 1\le i\le n, 1\le j\le m$. Thus

$$\begin{aligned}(a+b)\theta_x &= (t_1+\cdots+t_n+k_1+\cdots+k_m)\theta_x\\ &= x(t_1+\cdots+t_n+k_1+\cdots+k_m)\\ &= t_{1x}+\cdots+t_{nx}+k_{1x}+\cdots k_{mx}\\ &= (t_1+\cdots+t_n)\theta_x+(k_1+\cdots+k_m)\theta_x\\ &= (a)\theta_x+(b)\theta_x.\end{aligned}$$

Hence θ_x is a homomorphism.

Now if $(a)\theta_x=(b)\theta_x$ then

$$\begin{aligned}t_{1x}+\cdots+t_{nx} &= k_{1x}+\cdots+k_{mx}\\ t_1+\cdots+t_n &= k_1+\cdots+k_m.\end{aligned}$$

Thus $a=b$ and θ_x is a monomorphism.

Finally if $t_{1x}+\cdots+t_{nx}$ is an element of $\mathrm{sg}< t_x;t\in T>$, then as $xt_i=t_{ix}$, $1\le i\le n$, we have

$$t_{1x}+\cdots+t_{nx}=xt_1+\cdots+xt_n=x(t_1+\cdots+t_n),$$

where $t_1+\cdots+t_n\in(\mathrm{Frs}_{\mathcal{V}}(T),+)$. So $(t_1+\cdots+t_n)\theta_x=t_{1x}+\cdots+t_{nx}$ which proves that θ_x is an epimorphism and we have shown that

$$(\mathrm{Frs}_{\mathcal{V}}(T),+)\cong \mathrm{sg}< t_x;t\in T>.$$

The right hand side of the above isomorphism is the free semigroup in $\mathcal{V}$ on the set $\{t_x; t \in T\}$ in which for each t_x there corresponds an element $t \in T$. This implies that $(\mathrm{Frs}_{\mathcal{V}}(T), +)$ is the free semigroup in $\mathcal{V}$ on the set T.

(ii) Let H be a T-semigroup in $\mathcal{V}$; then there exists a homomorphism θ where

$$\theta : T \longrightarrow \mathrm{End}(H).$$

As $H \in \mathcal{V}$, $E(H)$ also lies in $\mathcal{V}$ by theorem 2.2.8. Since $(\mathrm{Frs}_{\mathcal{V}}(T), +)$ is the free semigroup in $\mathcal{V}$ on T and $E(H) \in \mathcal{V}$, we can extend θ to a semigroup homomorphism ψ such that

$$\psi : (\mathrm{Frs}_{\mathcal{V}}(T), +) \longrightarrow (E(H), +).$$

By lemma 2.2.5, ψ is a d.g. homomorphism which means that ψ is a d.g. representation of $(\mathrm{Frs}_{\mathcal{V}}(T), T)$ on H or equivalently H is a $(\mathrm{Frs}_{\mathcal{V}}(T), T)$-semigroup.

(iii) Since $(\mathrm{Frs}_{\mathcal{V}}(T), +)$ is the free semigroup in $\mathcal{V}$ on the set T and $(S, +)$ lies in $\mathcal{V}$, then θ can be extended uniquely to a semigroup homomorphism

$$\eta : (\mathrm{Frs}_{\mathcal{V}}(T), +) \longrightarrow (S, +).$$

By lemma 2.2.5, η is a d.g. seminear-ring homomorphism from $(\mathrm{Frs}_{\mathcal{V}}(T), T)$ to (S, U).

Remark: It should be noticed that the structure of $(\mathrm{Frs}_{\mathcal{V}}(T), T)$ is independent of the set X. This fact is clear from theorem 2.3.3(i).

Corollary 2.3.4 Let (S, T) be a d.g. seminear-ring in $\mathcal{V}$. Then there is a d.g. epimorphism $\theta : (\mathrm{Frs}_{\mathcal{V}}(T), T) \longrightarrow (S, T)$ extending the identity map on T.

In order to proceed and give the next result, we need to define the free (S, T)-semigroup in a given variety $\mathcal{V}$. So we recall definition 2.2.11 and state the following.

Definition 2.3.5 Let (S, T) be a d.g. seminear-ring in a variety $\mathcal{V}$. An (S, T)-semigroup H is a free (S, T)-semigroup in $\mathcal{V}$ on a set X if $H \in \mathcal{V}$, $X \subseteq H$ and if K is an (S,T)-semigroup in $\mathcal{V}$ then any map $\theta : X \longrightarrow K$ can be extended uniquely to an (S,T) homomorphism from H to K.

The following theorem tells us about the free (S,T)-semigroup that we have already met.

Theorem 2.3.6 The free semigroup $\mathrm{Frs}_{\mathcal{V}}(X,T)$ in a variety $\mathcal{V}$ on the set T_X, is the free $(\mathrm{Frs}_{\mathcal{V}}(T),T)$-semigroup in $\mathcal{V}$ on the set X.

Proof Let H be a $(\mathrm{Frs}_{\mathcal{V}}(T),T)$-semigroup in $\mathcal{V}$. Let θ be a map such that

$$\theta : X \longrightarrow H.$$

Then we can extend θ in a unique way to be a T-homomorphism

$$\theta : \mathrm{Frs}_{\mathcal{V}}(X,T) \longrightarrow H$$

as the following shows.

Since we already have $x.t = t_x$, $x \in X$, $t \in T$, it follows that

$$(t_x)\theta = (x.t)\theta = (x\theta)t.$$

Hence θ is extended (as a map) to T_X.

Since $\mathrm{Frs}_{\mathcal{V}}(X,T) = \mathrm{sg}< T_X >$ and $\mathrm{Frs}_{\mathcal{V}}(X,T)$ is a free semigroup on the set T_X , therefore θ is indeed extended uniquely to be a homomorphism from $\mathrm{Frs}_{\mathcal{V}}(X,T)$ to H.

Now it only remains to show that θ is an (S,T)-homomorphism.

Let $y \in \mathrm{Frs}_{\mathcal{V}}(X,T)$, then $y = t_{1x_1} + \cdots + t_{nx_n}$, where $t_i \in T \bigcup \{1\}$,

$x_i \in X$, $1 \leq i \leq n$, and 1_{x_i} represents x_i.

For $t \in T$, we have

$$\begin{aligned}
(yt)\theta &= ((t_{1x_1} + \cdots + t_{nx_n})t)\theta \\
&= (t_{1x_1}t + \cdots + t_{nx_n}t)\theta \\
&= ((t_1t)_{x_1} + \cdots + (t_nt)_{x_n})\theta \\
&= (t_1t)_{x_1}\theta + \cdots + (t_nt)_{x_n}\theta \\
&= (x_1\theta)t_1t + \cdots + (x_n\theta)t_nt \\
&= ((x_1\theta)t_1 + \cdots + (x_n\theta)t_n)t
\end{aligned}$$

$$
\begin{aligned}
&= ((x_1t_1)\theta + \cdots + (x_nt_n)\theta)t \\
&= ((t_{1x_1})\theta + \cdots + (t_{nx_n})\theta)t \\
&= (t_{1x_1} + \cdots + t_{nx_n})\theta t \\
&= (y\theta)t.
\end{aligned}
$$

That is θ is a T-homomorphism and the result is proved.

Now we are able to prove the existence of the free (S,T)-semigroup in $\mathcal{V}$ on a set X .

Theorem 2.3.7 The free (S,T)-semigroup in $\mathcal{V}$ on a set X always exists.

Proof. Let (S,T) be a d.g. seminear-ring; then by corollary 2.3.4, there is a d.g. epimorphism θ which extends the identity map on T such that

$$\theta : (\mathrm{Frs}_{\mathcal{V}}(T), T) \longrightarrow (S,T),$$

where $(\mathrm{Frs}_{\mathcal{V}}(T), T)$ is the free d.g. seminear-ring on T in $\mathcal{V}$.

Let $\rho := \ker\ \theta$. Consider $\mathrm{Frs}_{\mathcal{V}}(X,T)$ and let N be the least congruence in $\mathrm{Frs}_{\mathcal{V}}(X,T)$ containing $\mathrm{Frs}_{\mathcal{V}}(X,T)\rho$, where

$$\mathrm{Frs}_{\mathcal{V}}(X,T)\rho = \{(ky_1, ky_2) : k \in \mathrm{Frs}_{\mathcal{V}}(X,T), (y_1, y_2) \in \rho\}.$$

Let

$$H := \mathrm{Frs}_{\mathcal{V}}(X,T)/N.$$

We show that H is the free (S,T)-semigroup in $\mathcal{V}$ on a set X.

Certainly H is an $(\mathrm{Frs}_{\mathcal{V}}(T), T)$-semigroup. From above, the kernel of the representation of $(\mathrm{Frs}_{\mathcal{V}}(T), T)$ contains ρ. Thus as $(S,T) \cong (\mathrm{Frs}_{\mathcal{V}}(T), T)/\rho$, we can define H canonically as an (S,T)-semigroup. Let K be an (S,T)-semigroup. Then it is a T-semigroup and, by theorem 2.3.3(ii), it is an $(\mathrm{Frs}_{\mathcal{V}}(T), T)$-semigroup.

Since K is an (S,T)-semigroup, the representation of $(\mathrm{Frs}_{\mathcal{V}}(T), T)$ on K contains ρ in its kernel.

Let ϕ be a map such that $\phi : X \longrightarrow K$. Then by theorem 2.3.6, ϕ can be extended to an $(\mathrm{Frs}_{\mathcal{V}}(T), T)$-homomorphism

$$\phi : \mathrm{Frs}_{\mathcal{V}}(X,T) \longrightarrow K.$$

Let $k \in \mathrm{Frs}_{\mathcal{V}}(X,T)$, $(y_1, y_2) \in \rho$, then

$$k(y_1, y_2) = (ky_1, ky_2) \in \ker\phi,$$

since the representation of $(\mathrm{Frs}_{\mathcal{V}}(T), T)$ on K contains ρ in its kernel.

Thus $N \subseteq \ker\phi$, N being the least congruence in $\mathrm{Frs}_{\mathcal{V}}(X,T)$ containing $\mathrm{Frs}_{\mathcal{V}}(X,T)\rho$.

This means that ϕ can be factored as a product $\psi\phi'$ where ψ is the natural $(\mathrm{Frs}_{\mathcal{V}}(T), T)$-homomorphism such that

$$\psi : \mathrm{Frs}_{\mathcal{V}}(X,T) \longrightarrow H.$$

We deduce that ϕ' is a T-homomorphism from the (S,T)-semigroup H to K extending ϕ. Since ϕ' agrees with ϕ on the T-generating set X of H, it is uniquely defined, forcing H to be the free (S,T)-semigroup in $\mathcal{V}$ on the set X.

In order to proceed to the next result, we have to state the following definition.

Definition 2.3.8 Let $\{H_\lambda; \lambda \in \Lambda\}$ be a set of (S,T)-semigroups in a variety $\mathcal{V}$. A semigroup H is the free (S,T)-product of the set $\{H_\lambda; \lambda \in \Lambda\}$ in $\mathcal{V}$ if for each $\lambda \in \Lambda$, there exists an (S,T)-homomorphism $\theta_\lambda : H_\lambda \longrightarrow H$, and if K is an (S,T)-semigroup in $\mathcal{V}$ with the property that for each $\lambda \in \Lambda$, there exists an (S,T)-homomorphism $\phi_\lambda : H_\lambda \longrightarrow K$, then there exists a unique (S,T)-homomorphism $\psi : H \longrightarrow K$ such that $\phi_\lambda = \theta_\lambda\psi$.

The next result in this section will assure the existence of the free (S,T)-product of a set of (S,T)-semigroups.

Theorem 2.3.9 The free (S,T)-product of the set of (S,T)-semigroups $\{H_\lambda; \lambda \in \Lambda\}$ in a variety $\mathcal{V}$ exists.

Proof We will assume that all semigroups which we are considering lie in the variety $\mathcal{V}$ where $\mathcal{V}$ is a fixed variety. Let $\{H_\lambda; \lambda \in \Lambda\}$ be a set of (S,T)-semigroups in $\mathcal{V}$. For each $\lambda \in \Lambda$, let X_λ be an (S,T)-generating set for H_λ.

Define

$$X := \bigcup_{\lambda\in\Lambda} X_\lambda$$

to be the disjoint union of the X_λs.

Without loss of generality, we assume that X_λ is a subset of X for each λ.

Define

$$F_\lambda := \mathrm{Frs}_\mathcal{V}(X_\lambda^{f_\lambda}, S, T) \text{ for each } \lambda \in \Lambda,$$

that is the free (S,T)-semigroup on X_λ, where $X_\lambda^{f_\lambda}$ is a copy of X_λ.

Also define

$$F := \mathrm{Frs}_\mathcal{V}(X^f, S, T),$$

the free (S,T)-semigroup on X, where X^f is a copy of X.

Let $F'_\lambda \subseteq F$ be the (S,T)-subsemigroup of F generated by X_λ^f regarded as a subset of X^f.

Consider the identity map

$$i : X_\lambda^{f_\lambda} \longrightarrow X_\lambda^f,$$

where $X_\lambda^{f_\lambda} \subseteq F_\lambda$ and $\quad X_\lambda^f \subseteq F$.

Since F_λ is the free (S,T)-semigroup on $X_\lambda^{f_\lambda}$, the above map extends to an (S,T)-epimorphism

$$\theta : F_\lambda \longrightarrow F'_\lambda.$$

Conversely, if the map

$$\bar{\theta} : X^f \longrightarrow F_\lambda$$

is given by

$$(X_\mu)\bar{\theta} = \begin{cases} X_\lambda^{f_\lambda} & \text{if } \mu = \lambda\,, \\ 0 & \text{otherwise} \end{cases}$$

then $\bar{\theta}$ extends to an (S,T)-homomorphism

$$\phi : F \longrightarrow F_\lambda.$$

If we restricted ϕ to $F'_\lambda \subseteq F$ and called it again ϕ, then

$$\phi : F'_\lambda \longrightarrow F_\lambda$$

is an (S,T)-epimorphism.

We deduce that $\theta\phi = \phi\theta$ are the identity maps on their respective X_λ s,

hence are identity maps on F_λ and F'_λ, respectively.

Thus F_λ is (S,T)-isomorphic to F'_λ and we may assume that $F_\lambda \subseteq F$ for each $\lambda \in \Lambda$.

Consider now the identity map

$$\bar{\imath} : X_\lambda^{f_\lambda} \longrightarrow X_\lambda ,$$

where $X_\lambda^{f_\lambda} \subseteq F_\lambda$, then it can be extended to an (S,T)-homomorphism

$$\theta_\lambda : F_\lambda \longrightarrow H_\lambda.$$

Let $k_\lambda = \ker \theta_\lambda$, then

$$H_\lambda \cong F_\lambda / k_\lambda.$$

Define k to be the congruence of F generated by the set $\{k_\lambda; \lambda \in \Lambda \}$, then

$$k \bigcap (F_\lambda \times F_\lambda) \supseteq k_\lambda.$$

Let

$$H := F/k.$$

We claim that H is the free (S,T)-product of the set of (S,T)-semigroups $\{H_\lambda; \lambda \in \Lambda\}$.

Since $k \bigcap (F_\lambda \times F_\lambda) \supseteq k_\lambda$, it follows that $F_\lambda / k \bigcap (F_\lambda \times F_\lambda)$ is an (S,T)-homomorphic image of F_λ / k_λ.

Furthermore, $F_\lambda / k \bigcap (F_\lambda \times F_\lambda) \cong (F_\lambda + k)/k$.

Combining again the above homomorphisms, we have

$$(F_\lambda + k)/k \cong F_\lambda / k \bigcap (F_\lambda \times F_\lambda)$$

which is a homomorphic image of $F_\lambda / k_\lambda \cong H_\lambda$.

Thus we obtain an (S,T)-homomorphism α_λ where

$\alpha_\lambda : H_\lambda \longrightarrow (F_\lambda + k)/k$ is given successively by

$$H_\lambda \longrightarrow F_\lambda / k_\lambda \longrightarrow F_\lambda / k \bigcap (F_\lambda \times F_\lambda) \longrightarrow (F_\lambda + k)/k,$$

and we can see that for $x_\lambda \in X_\lambda, (x_\lambda)\alpha_\lambda = x^f + k$.

Let P be an (S,T)-semigroup. For each $\lambda \in \Lambda$, let

$\phi_\lambda : H_\lambda \longrightarrow P$ be an (S,T)-homomorphism.

For each $\lambda \in \Lambda$, where $x_\lambda^f \in F'_\lambda \subseteq F$, define a map

$\theta : F'_\lambda \longrightarrow P$ by

$$x^f_\lambda \theta = x_\lambda \phi_\lambda, \text{ where } x_\lambda \in X_\lambda \subseteq H_\lambda.$$

Since F is the free (S,T)-semigroup on the set $X = \bigcup_{\lambda \in \Lambda} X_\lambda$, θ can be extended to be an (S,T)-homomorphism

$\theta : F \longrightarrow P$ by defining

$$x^f \theta = x\phi_\lambda, \text{ for each } \lambda \in \Lambda,$$

where $x \in X_\lambda$ and x^f belongs to some X^f_λ whose ϕ_λ is to be considered.

Now F_λ is generated as an (S,T)-semigroup by $X^{f_\lambda}_\lambda$, and by definition we have

$$X^{f_\lambda}_\lambda \theta = X_\lambda \phi_\lambda.$$

Thus

$$F_\lambda \theta = H_\lambda \phi_\lambda \text{ for each } \lambda \in \Lambda.$$

Since $H_\lambda \cong F_\lambda / k_\lambda$, each $x_\lambda \in H_\lambda$ is mapped to $x^{f_\lambda} + k_\lambda$.
Hence

$$\ker \theta \supseteq k_\lambda \text{ for each } \lambda \in \Lambda.$$

We deduce that ker θ is a congruence which contains k_λ for all $\lambda \in \Lambda$ and hence

$$\ker \theta \supseteq k.$$

This means that we can define an (S,T)-homomorphism

$\phi : H \longrightarrow P$ by

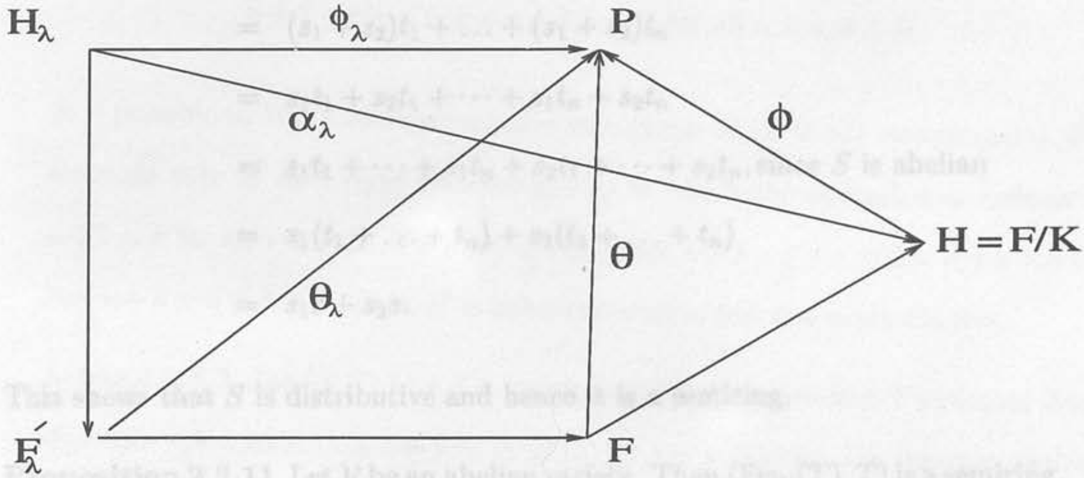
$$(f + k)\phi := f\theta \text{ for all } f \in F.$$

The next step is to show that $\alpha_\lambda \phi = \phi_\lambda$ for each $\lambda \in \Lambda$.

For each $\lambda \in \Lambda$, we have

$$x_\lambda \alpha_\lambda \phi = (x^f + k)\phi = (x^f)\theta = x_\lambda \phi_\lambda, \text{ for all } x_\lambda \in X_\lambda.$$

Since $\alpha_\lambda \phi$ and ϕ_λ are (S,T)-homomorphisms and X_λ generates H_λ as an (S,T)-semigroup, this shows that



$$\alpha_\lambda \phi = \phi_\lambda \text{ for all } \lambda \in \Lambda.$$

Finally we show that ϕ is unique.

Suppose that $\psi : H \longrightarrow P$ is another (S,T)-homomorphism satisfying

$$\alpha_\lambda \psi = \phi_\lambda \text{ for each } \lambda \in \Lambda.$$

Then as $x_\lambda \alpha_\lambda = x^f + k$, we have

$$(x^f + k)\psi = (x_\lambda \alpha_\lambda)\psi = x_\lambda \phi_\lambda = x^f \theta = (x^f + k)\phi,$$

for each $x^f \in X_\lambda^f, \lambda \in \Lambda$.

So ψ and ϕ are both (S,T)-homomorphisms which agree on the (S,T)-generating set X and hence $\psi = \phi$ and ϕ is unique.

This completes the proof for H to be the free (S,T)-product of the set of (S,T)-semigroups $\{H_\lambda; \lambda \in \Lambda\}$ in the variety $\mathcal{V}$.

Before moving to the next section we give the following two propositions.

Proposition 2.3.10 Let (S,T) be a d.g. seminear-ring which is abelian. Then S is a semiring.

Proof Let s, s_1, $s_2 \in S$. Then we may write

$$s = t_1 + \ldots + t_n,\ t_i \in T,\ 1 \le i \le n. \text{ So}$$

$$(s_1 + s_2)s = (s_1 + s_2)(t_1 + \ldots + t_n)$$

$$\begin{aligned}
&= (s_1+s_2)t_1+\ldots+(s_1+s_2)t_n \\
&= s_1t_1+s_2t_1+\cdots+s_1t_n+s_2t_n \\
&= s_1t_1+\cdots+s_1t_n+s_2t_1+\cdots+s_2t_n, \text{since } S \text{ is abelian} \\
&= s_1(t_1+\ldots+t_n)+s_2(t_1+\ldots+t_n) \\
&= s_1s+s_2s.
\end{aligned}$$

This shows that S is distributive and hence it is a semiring.

Proposition 2.3.11 Let $\mathcal{V}$ be an abelian variety. Then $(\mathrm{Frs}_{\mathcal{V}}(T), T)$ is a semiring.

Proof Recall that $(\mathrm{Frs}_{\mathcal{V}}(T), T)$ has a faithful d.g. representation on $\mathrm{Frs}_{\mathcal{V}}(X, T) = H$, say. Then $T \subseteq \mathrm{End}(H)$. Since H belongs to the abelian variety $\mathcal{V}$, $\mathrm{End}(H)$ is a semiring by proposition 2.3.10. Thus all sums of elements from T lie in $\mathrm{End}(H)$, which means that $\mathrm{Frs}_{\mathcal{V}}(T)$ is embedded in $\mathrm{End}(H)$. Hence $(\mathrm{Frs}_{\mathcal{V}}(T), T)$ is a semiring.

2.4 D.g. representation

We have already seen in chapter 1 that every seminear-ring has a faithful representation. Our main object in this section is to show that this situation is not always true in the case of d.g. seminear-rings. Thus given a d.g. seminear-ring (S, T) then it is not possible in general to find a semigroup H such that (S, T) is embedded in $(E(H)$,$\mathrm{End}(H))$ with T embedded in $\mathrm{End}(H)$, in other words, "not every d.g. seminear-ring has a faithful d.g. representation", as the following main theorem says.

Theorem 2.4.1 There exist d.g. seminear-rings in a variety $\mathcal{V}$ which do not have a faithful d.g. representation.

To prove this result we need to state the following definition and one lemma regarding this definition.

Definition 2.4.2 A semigroup $(H, +)$ is said to be subcommutative if it satisfies

$$a+b+c+d = a+c+b+d \text{ , for all } a,b,c,d \in H.$$

It is possible to find a subcommutative semigroup which is not commutative. For example take H as a right zero semigroup in which the operation is defined as $a+b=b$ for all $a,b \in H$. Then we can see that $a+b+c+d = d = a+c+b+d$ but $a+b=b \neq b+a$. Thus H is subcommutative but not commutative.

Lemma 2.4.3 Let (S,T) be a d.g. seminear-ring. Suppose that T contains three elements a,b,c such that $a+b=c$. If (S,T) has a faithful d.g. representation, then a and b are relatively subcommutative in the sense of the relation :

$$a+b+a+b = a+a+b+b \ .$$

Proof Let (S,T) be a d.g. seminear-ring having a faithful d.g. representation on a semigroup H. Then for all $h \in H$, we have

$$\begin{aligned}(h+h)c &= hc+hc, \text{ as } c \in T \text{ is mapped to an endomorphism of } H,\\ &= h(c+c)\\ &= h(a+b+a+b).\end{aligned}$$

On the other hand, we have

$$\begin{aligned}(h+h)c &= (h+h)(a+b)\\ &= (h+h)a+(h+h)b\\ &= ha+ha+hb+hb, \text{ as } a,b \in T \text{ are mapped to End}(H),\\ &= h(a+a+b+b).\end{aligned}$$

Thus

$$h(a+b+a+b) = h(a+a+b+b).$$

Since H is a faithful S-module, we conclude that

$$a+b+a+b = a+a+b+b.$$

Now we are able to prove theorem 2.4.1.

Proof of theorem 2.4.1 Let $\mathcal{V}$ be a variety of semigroups not all of which are subcommutative. Let $T = \{x,y,z,0\}$ be a semigroup with all products equal to zero. Let $(\mathrm{Frs}_{\mathcal{V}}(T),T)$ be the free d.g. seminear-ring on T in $\mathcal{V}$. Then all products in $\mathrm{Frs}_{\mathcal{V}}(T)$ are zero, and hence a semigroup congruence on $(\mathrm{Frs}(T), +)$ is a seminear-ring congruence on $(\mathrm{Frs}_{\mathcal{V}}(T), +, .)$. Let ρ be the least congruence on $\mathrm{Frs}_{\mathcal{V}}(T)$ containing $(x+y, z)$. Define (S,T) to be $(\mathrm{Frs}_{\mathcal{V}}(T), T)/\rho$, the canonical homomorphic image of $\mathrm{Frs}_{\mathcal{V}}(T)$ by ρ.

We show that S is the free semigroup in $\mathcal{V}$ on two generators, namely, $x+\rho$ and $y+\rho$.

Let Q be a free semigroup in $\mathcal{V}$ on two generators $x'$and y', say.

Consider the map $\phi : T \longrightarrow \{x', y'\}$ given by

$$(x)\phi = x' \ , \ (y)\phi = y' \ , \ (z)\phi = x' + y' \ .$$

Then ϕ extends to be a homomorphism $\phi^{\star}$ from $(\mathrm{Frs}_{\mathcal{V}}(T), +)$ to $(Q, +)$ with ker $\phi^{\star} \supseteq \{(x+y, z)\}$. Hence ker $\phi^{\star} \supseteq \rho$, and $\phi^{\star}$ can be factored through $(S, +)$. In other words there exists a homomorphism ψ from $(S, +)$ onto $(Q, +)$ such that

$$(x+\rho)\psi = x' \text{ and } (y+\rho)\psi = y'.$$

This shows that $(S, +)$ is free in $\mathcal{V}$ on $\{x+\rho, y+\rho\}$. Letting $a = x+\rho, b = y+\rho$, we have

$$a + b = (x+\rho) + (y+\rho) = z + \rho = c, \text{ say.}$$

By lemma 2.4.3, if (S,T) has a faithful d.g. representation, then a and b are relatively subcommutative. But by the hypothesis, two free generators of a free semigroup in $\mathcal{V}$ cannot be relatively subcommutative. This implies that (S,T) cannot have a faithful d.g. representation and the theorem is proved.

Out of chapter 2 a research paper entitled "On Free d.g. Seminear-rings" has been accepted for publication. Two more papers based on the results in chapters 5 to 10 are in preparation.

$$
\begin{tikzcd}
T \arrow[rr, "\phi"] \arrow[d, bend right] & & \{x,y\} \subseteq (Q,+) \\
Frs_{\upsilon}((T),+) \arrow[rr] \arrow[urr, "\phi^{*}"] & & (S,+) \arrow[u, bend right]
\end{tikzcd}
$$

Chapter 3

Strong semilattice of near-rings and rings

3.1 Preliminaries

An element e of a semigroup S is called an idempotent if $e^2 = e$. So 0 and 1, if they are in S, are idempotents. A lower semilattice is a partially ordered set in which each pair of elements has a greatest lower bound. It is well-known that if $(E, \leq)$ is a lower semilattice, then $(E, \wedge)$ is a commutative semigroup of idempotents and for all $a, b \in E$, $a \leq b$ if and only if $a \wedge b = a$. Conversely, let $(E, .)$ be a commutative semigroup of idempotents. Then the relation $\leq$ on E defined by $a \leq b$ if and only if $a.b = a$ is a partial order on E such that $(E, \leq)$ is a lower semilattice. This means that the notions of commutative semigroup of idempotents and lower semilattice are equivalent. So the term semilattice can be used with either meaning.

Definition 3.1.1 Let S be a semigroup. S is called a strong semilattice of groups (see Howie [4]) if there exists a semilattice Y such that

1) $\{G_\alpha : \alpha \in Y\}$ is a family of disjoint subgroups of S indexed by Y;

2) for each pair $\alpha, \beta \in Y$ with $\alpha \geq \beta$ there exists a homomorphism $\phi_{\alpha,\beta} : G_\alpha \longrightarrow G_\beta$ such that

(i) $\phi_{\alpha,\alpha}$ is the identical automorphism of S_α for each $\alpha \in Y$;

(ii) $\phi_{\alpha,\beta}\phi_{\beta,\gamma} = \phi_{\alpha,\gamma}$ for every $\alpha, \beta, \gamma \in Y$ such that $\alpha \geq \beta \geq \gamma$.

The above semilattice is denoted by:

$$S = (Y, \{G_\alpha\}_{\alpha\in Y}, \{\phi_{\alpha,\beta}\}_{\alpha\geq\beta}).$$

3.2 Special classes of semigroups

As there is a strong connection between semilattices of groups and some special kinds of semigroups, we present a quick view of the definitions of those semigroups which will be involved in the forthcoming chapters.

Definition 3.2.1 An element a of a semigroup S is called regular if there exists b in S such that $aba = a$. The semigroup S is called regular if all its elements are regular.

Definition 3.2.2 In a semigroup, an element b is called an inverse of a if $aba = a$, and $bab = b$. A semigroup S is called an inverse semigroup if every element of S has a unique inverse. Equivalently, S is an inverse semigroup if S is a regular semigroup whose idempotents commute.

A special class of inverse semigroups is the following.

Definition 3.2.3 A semigroup which is a semilattice of groups is called a Clifford semigroup. It is known that a Clifford semigroup is an inverse semigroup whose idempotents lie in the centre.

3.3 Strong semilattice of near-rings

In this section we will start with a family of near-rings each of which corresponds to a member of a semilattice Y. Using a construction method obtained from definition 3.1.1, we proceed to look at a strong semilattice of near-rings which turns out to be a seminear-ring.

Let Y be a semilattice and let $\{S_\alpha; \alpha\in Y\}$ be a disjoint family of near-rings. S is called a strong semilattice of the near-rings $\{S_\alpha; \alpha\in Y\}$ if for each pair α, β of elements of Y with $\alpha\geq\beta$, there exists a homomorphism $\phi_{\alpha,\beta} : S_\alpha \longrightarrow S_\beta$ such that

(i) $\phi_{\alpha,\alpha}$ is the identical automorphism of S_α for each $\alpha \in Y$;

(ii) $\phi_{\alpha,\beta}\phi_{\beta,\gamma} = \phi_{\alpha,\gamma}$ for every $\alpha, \beta, \gamma \in Y$ such that $\alpha \geq \beta \geq \gamma$.

Let $S = \bigcup\{S_\alpha; \alpha \in Y\}$, and define two operations $\oplus$ and $\odot$ on S as follows

$$\begin{aligned} a_\alpha \oplus b_\beta &= a_\alpha\phi_{\alpha,\alpha\beta} + b_\beta\phi_{\beta,\alpha\beta}; \\ a_\alpha \odot b_\beta &= (a_\alpha\phi_{\alpha,\alpha\beta})(b_\beta\phi_{\beta,\alpha\beta}). \end{aligned}$$

Observe that $\alpha\beta$ denotes the product of α and β in the semilattice Y, while the addition and multiplication of $a_\alpha\phi_{\alpha,\alpha\beta}$ and $b_\beta)\phi_{\beta,\alpha\beta}$ take place in the near-ring $S_{\alpha\beta}$.
We show that $(S, \oplus)$ is a semigroup.
Let $a_\alpha \in S_\alpha, b_\beta \in S_\beta, c_\gamma \in S_\gamma$ and write $\alpha\beta = \delta, \beta\gamma = \epsilon, \alpha\beta\gamma = \eta$ in Y, then

$$\begin{aligned} (a_\alpha \oplus b_\beta) \oplus c_\gamma &= (a_\alpha\phi_{\alpha,\alpha\beta} + b_\beta\phi_{\beta,\alpha\beta}) \oplus c_\gamma \\ &= (a_\alpha\phi_{\alpha,\delta} + b_\beta\phi_{\beta,\delta}) \oplus c_\gamma \\ &= (a_\alpha\phi_{\alpha,\delta} + b_\beta\phi_{\beta,\delta})\phi_{\delta,\delta\gamma} + c_\gamma\phi_{\gamma,\delta\gamma} \\ &= (a_\alpha\phi_{\alpha,\delta} + b_\beta\phi_{\beta,\delta})\phi_{\delta,\eta} + c_\gamma\phi_{\gamma,\eta} \\ &= a_\alpha\phi_{\alpha,\delta}\phi_{\delta,\eta} + b_\beta\phi_{\beta,\delta}\phi_{\delta,\eta} + c_\gamma\phi_{\gamma,\eta} \\ &= a_\alpha\phi_{\alpha,\eta} + b_\beta\phi_{\beta,\eta} + c_\gamma\phi_{\gamma,\eta} \end{aligned}$$

Similarly we have

$$\begin{aligned} a_\alpha \oplus (b_\beta \oplus c_\gamma) &= a_\alpha \oplus (b_\beta\phi_{\beta,\beta\gamma} + c_\gamma\phi_{\gamma,\beta\gamma}) \\ &= a_\alpha \oplus (b_\beta\phi_{\beta,\epsilon} + c_\gamma\phi_{\gamma,\epsilon}) \\ &= a_\alpha\phi_{\alpha,\alpha\epsilon} + (b_\beta\phi_{\beta,\epsilon} + c_\gamma\phi_{\gamma,\epsilon})\phi_{\epsilon,\alpha\epsilon} \\ &= a_\alpha\phi_{\alpha,\alpha\epsilon} + b_\beta\phi_{\beta,\epsilon}\phi_{\epsilon,\alpha\epsilon} + c_\gamma\phi_{\gamma,\epsilon}\phi_{\epsilon,\alpha\epsilon} \\ &= a_\alpha\phi_{\alpha,\eta} + b_\beta\phi_{\beta,\eta} + c_\gamma\phi_{\gamma,\eta} \end{aligned}$$

Thus

$$(a_\alpha \oplus b_\beta) \oplus c_\gamma = a_\alpha \oplus (b_\beta \oplus c_\gamma),$$

and $(S, \oplus)$ is a semigroup.
Now consider the operation $\odot$ defined on S, then we have

$$(a_\alpha \odot b_\beta) \odot c_\gamma = [(a_\alpha\phi_{\alpha,\alpha\beta})(b_\beta\phi_{\beta,\alpha\beta})] \odot c_\gamma$$

$$
\begin{aligned}
&= [(a_\alpha\phi_{\alpha,\delta})(b_\beta\phi_{\beta,\delta})] \odot c_\gamma \\
&= [(a_\alpha\phi_{\alpha,\delta})(b_\beta\phi_{\beta,\delta})]\phi_{\delta,\delta\gamma}(c_\gamma\phi_{\gamma,\delta\gamma}) \\
&= [(a_\alpha\phi_{\alpha,\delta})(b_\beta\phi_{\beta,\delta})]\phi_{\delta,\eta}(c_\gamma\phi_{\gamma,\eta}) \\
&= (a_\alpha\phi_{\alpha,\delta}\phi_{\delta,\eta})(b_\beta\phi_{\beta,\delta}\phi_{\delta,\eta})(c_\gamma\phi_{\gamma,\eta}) \\
&= (a_\alpha\phi_{\alpha,\eta})(b_\beta\phi_{\beta,\eta})(c_\gamma\phi_{\gamma,\eta}).
\end{aligned}
$$

Similarly we can get

$$a_\alpha \odot (b_\beta \odot c_\gamma) = (a_\alpha\phi_{\alpha,\eta})(b_\beta\phi_{\beta,\eta})(c_\gamma\phi_{\gamma,\eta}).$$

Hence

$$(a_\alpha \odot b_\beta) \odot c_\gamma = a_\alpha \odot (b_\beta \odot c_\gamma)$$

and $(S, \odot)$ is a semigroup.

Finally, we can see that

$$
\begin{aligned}
c_\gamma \odot (a_\alpha \oplus b_\beta) &= c_\gamma \odot (a_\alpha\phi_{\alpha,\delta} + b_\beta\phi_{\beta,\delta}) \\
&= (c_\gamma\phi_{\gamma,\gamma\delta})((a_\alpha\phi_{\alpha,\delta} + b_\beta\phi_{\beta,\delta})\phi_{\delta,\gamma\delta}) \\
&= (c_\gamma\phi_{\gamma,\gamma\delta})(a_\alpha\phi_{\alpha,\delta}\phi_{\delta,\gamma\delta} + b_\beta\phi_{\beta,\delta}\phi_{\delta,\gamma\delta}) \\
&= (c_\gamma\phi_{\gamma,\gamma\delta})(a_\alpha\phi_{\alpha,\gamma\delta} + b_\beta\phi_{\beta,\gamma\delta}) \\
&= (c_\gamma\phi_{\gamma,\eta})(a_\alpha\phi_{\alpha,\eta} + b_\beta\phi_{\beta,\eta}) \\
&= (c_\gamma\phi_{\gamma,\eta})(a_\alpha\phi_{\alpha,\eta}) + (c_\gamma\phi_{\gamma,\eta})(b_\beta\phi_{\beta,\eta}).
\end{aligned}
$$

Also we have

$$
\begin{aligned}
(c_\gamma \odot a_\alpha) \oplus (c_\gamma \odot b_\beta) &= (c_\gamma\phi_{\gamma,\gamma\alpha})(a_\alpha\phi_{\alpha,\gamma\alpha}) \oplus (c_\gamma\phi_{\gamma,\gamma\beta})(b_\beta\phi_{\beta,\gamma\beta}) \\
&= (c_\gamma\phi_{\gamma,\omega})(a_\alpha\phi_{\alpha,\omega}) \oplus (c_\gamma\phi_{\gamma,\epsilon})(b_\beta\phi_{\beta,\epsilon})\text{ , where } \omega = \gamma\alpha, \\
&= [(c_\gamma\phi_{\gamma,\omega})(a_\alpha\phi_{\alpha,\omega})\phi_{\omega,\omega\epsilon}] + [(c_\gamma\phi_{\gamma,\epsilon})(b_\beta\phi_{\beta,\epsilon})\phi_{\epsilon,\omega\epsilon}] \\
&= [(c_\gamma\phi_{\gamma,\omega})(a_\alpha\phi_{\alpha,\omega})\phi_{\omega,\eta}] + [(c_\gamma\phi_{\gamma,\epsilon})(b_\beta\phi_{\beta,\epsilon})\phi_{\epsilon,\eta}] \\
&= (c_\gamma\phi_{\gamma,\omega}\phi_{\omega,\eta})(a_\alpha\phi_{\alpha,\omega}\phi_{\omega,\eta}) + (c_\gamma\phi_{\gamma,\epsilon}\phi_{\epsilon,\eta})(b_\beta\phi_{\beta,\epsilon}\phi_{\epsilon,\eta}) \\
&= (c_\gamma\phi_{\gamma,\eta})(a_\alpha\phi_{\alpha,\eta}) + (c_\gamma\phi_{\gamma,\eta})(b_\beta\phi_{\beta,\eta}).
\end{aligned}
$$

Hence

$$c_\gamma \odot (a_\alpha \oplus b_\beta) = (c_\gamma \odot a_\alpha) \oplus (c_\gamma \odot b_\beta)$$

and we have shown that S is a seminear-ring.

Thus a strong semilattice of near-rings is a seminear-ring.

Remark It is easy now to see that if we had started with a family of rings $\{R_\alpha; \alpha \in Y\}$ and followed the same way as above, then we could deduce that S is a semiring.

Chapter 4

Endomorphisms of inverse semigroups

This chapter, as stated in the abstract, is presented particularly as an introduction to the following chapters. In fact, it includes a collection of some results and basic ideas that will be involved, in addition to an outline of the main plan which will be followed in order to achieve our results. It should be mentioned that we will consider seminear-rings of endomorphisms of inverse semigroups in all the remaining chapters, where we are treating a special case in each chapter. So it may be appropriate to call those chapters : seminear-ring of endomorphisms I , seminear-ring of endomorphisms II , ... etc. In the following section we give a brief overview of our target.

4.1 An outline

We will consider semilattices of d.g. near-rings. Starting with some groups G_α where each α belongs to the semilattice Y, then we study the structure of the corresponding strong semilattice S. For each group G_α there will be a d.g. near-ring $E(G_\alpha)$ generated by $\mathrm{End}(G_\alpha)$, the set of all endomorphisms of G_α. On the other hand, considering S, the semilattice of the groups G_α, then End(S), the set of all endomorphisms of S, will generate a d.g. seminear-ring $E(S)$. So we study the structure of $E(S)$ with its relation to $\{E(G_\alpha)\,;\alpha\in Y\}$ which will lead to a Clifford semigroup. As mentioned above, a special case will be considered in each chapter depending on the groups and the relations assumed between these groups G_α. In each case (chapter) we may proceed along different ways of treating the object, but in the same pattern from the main plan. Although the cases are different, the final conclusions are similar. This would maintain the behaviour of the structure under consideration.

4.2 On endomorphisms of semilattices of groups

In this section we present some important and basic facts about the endomorphisms of semilattices of groups. We start with the following simple lemma.

Lemma 4.2.1 Let $S = (Y, \{G_\alpha\}_{\alpha\in Y}, \{\phi_{\alpha,\beta}\}_{\alpha\geq\beta})$ be a strong semilattice of groups. Then any subgroup of S has to be a subgroup of G_α for some $\alpha \in Y$. (i.e. S has no subgroups except the subgroups of G_α.)

Proof Suppose that H is a subgroup of S. Let e be the identity element of H, then $e = e_\alpha \in G_\alpha$ for some α. We show that H is a subgroup of G_α.

Let $h \in H$, then $h \in G_\beta$ for some β. Since e is the identity of $H, he = h$. But also we have

$$h = he = h\phi_{\beta,\alpha\beta}\, e\phi_{\alpha,\alpha\beta} \in G_{\alpha\beta}.$$

So $\beta = \alpha\beta$, which implies that $\beta \leq \alpha$.

Let h^{-1} be the inverse of h, then $h^{-1} \in G_\gamma$, say, for some γ.

Thus

$$hh^{-1} = e \in G_\alpha.$$

Also

$$hh^{-1} = h\phi_{\beta,\beta\gamma}\, h^{-1}\phi_{\gamma,\beta\gamma} \in G_{\beta\gamma}.$$

So $e \in G_{\beta\gamma}$ and $e \in G_\alpha$ which implies that $\alpha = \beta\gamma$ and $\alpha \leq \beta$.

Hence $\alpha = \beta$ and $h \in G_\alpha$.

This shows that an arbitrary element of H must lie in G_α, which forces H to be a subgroup of G_α.

Remark 1 Let f be an endomorphism of the strong semilattice S. If f is restricted on a group G_α then f is either an endomorphism of G_α or a homomorphism from G_α into a group G_ω for some $\omega \in Y$ such that $\omega = \alpha f$.

Now considering again the subgroups of S, we have

Theorem 4.2.2 Let $S = (Y, \{G_\alpha\}_{\alpha \in Y}, \{\phi_{\alpha,\beta}\}_{\alpha \geq \beta})$ be a strong semilattice of groups. If $f \in \mathrm{End}(S)$, then $G_\alpha f$ is a subgroup of S.

Proof Let $f \in \mathrm{End}(S)$. Consider $G_\alpha f \subseteq S$ and put $H_\alpha = G_\alpha f$.

Let $h_1, h_2 \in H_\alpha$. Then $h_1 = g_1 f$ and $h_2 = g_2 f$, for some $g_1, g_2 \in G_\alpha$.

Thus $h_2^{-1} = g_2^{-1} f$, which implies that

$$h_1 h_2^{-1} = (g_1 f)(g_2^{-1} f) = (g_1 g_2^{-1}) f \in H_\alpha.$$

Hence $G_\alpha f$ is a subgroup of S.

Remark 2 Let S be a strong semilattice of groups G_α. Let $\mathrm{IP}(S)$ be the set of all idempotent elements of S, (we ought to use the notation $E(S)$ for this purpose but since that notation is already reserved for the d.g. seminear-ring generated by the semigroup S, we depart from the usual one to use $\mathrm{IP}(S)$), then

$$\mathrm{IP}(S) = \{e_\alpha \,;\, \alpha \in Y\},$$

where e_α is the identity element of G_α for each α in the semilattice Y and hence

$$\mathrm{IP}(S) \cong Y.$$

It follows that an endomorphism $f \in \mathrm{End}(S)$ could be considered as acting as an endomorphism of Y.

Now we summarize some basic facts in the following theorem.

Theorem 4.2.3 Let $S = (Y, \{G_\alpha\}_{\alpha \in Y}, \{\phi_{\alpha,\beta}\}_{\alpha \geq \beta})$ be a strong semilattice of groups. Let $f \in \mathrm{End}(S)$; then the following hold:

(i) $G_\alpha f$ is a subgroup of S.

(ii) All subgroups of S are subgroups of the form $H \subseteq G_\alpha$ for some α.

(iii) $f|_{G_\alpha} \in \mathrm{End}(G_\alpha)$ or $f|_{G_\alpha} \in \mathrm{Hom}(G_\alpha, G_\omega)$ for some G_ω, where $\omega = \alpha f$.

(iv) $f|_Y$ is an endomorphism of Y.

We conclude this chapter by giving the following key result which is a vital tool in the coming work.

Theorem 4.2.4 Let $S = (Y, \{G_\alpha\}_{\alpha \in Y}, \{\phi_{\alpha,\beta}\}_{\alpha \geq \beta})$ be a strong semilattice of groups. Let $\alpha, \beta \in Y$ and let $g \in G_\alpha$. If $f \in \mathrm{End}(S)$, then

$$(g\phi_{\alpha,\alpha\beta})f = (gf)\phi_{\alpha f, \alpha f \beta f}.$$

Proof Let $g \in G_\alpha$, and suppose that e_β is the identity element of G_β. Then

$$ge_\beta = g\phi_{\alpha,\alpha\beta}e_\beta\phi_{\beta,\alpha\beta} = g\phi_{\alpha,\alpha\beta}e_{\alpha\beta} = g\phi_{\alpha,\alpha\beta}.$$

Let $f \in \mathrm{End}(S)$, then

$$(ge_\beta)f = (gf)(e_\beta f),$$

and from above, we have

$$(ge_\beta)f = (g\phi_{\alpha,\alpha\beta})f.$$

Suppose that $(e_\alpha)f = e_{\alpha'}$ and $(e_\beta)f = e_{\beta'}$,

where $e_{\alpha'} \in G_{\alpha'}$, $e_{\beta'} \in G_{\beta'}$, for some α', β' in which $\alpha f = \alpha'$, $\beta f = \beta'$.

Thus

$$(ge_\beta)f = (gf)(e_\beta f) = (gf)e_{\beta'}$$

Now $gf \in G_{\alpha'}$, since $g, e_\alpha \in G_\alpha$ with $e_\alpha f = e_{\alpha'} \in G_{\alpha'}$. So we have

$$(gf)e_{\beta'} = (gf)\phi_{\alpha',\alpha'\beta'}e_{\beta'}\phi_{\beta',\alpha'\beta'} = (gf)\phi_{\alpha',\alpha'\beta'}e_{\alpha'\beta'} = (gf)\phi_{\alpha',\alpha'\beta'}.$$

Hence

$$(g\phi_{\alpha,\alpha\beta})f = (ge_\beta)f = (gf)e_{\beta'} = (gf)\phi_{\alpha',\alpha'\beta'},$$

that is

$$(g\phi_{\alpha,\alpha\beta})f = (gf)\phi_{\alpha f,\alpha f\beta f}.$$

We note that if $\alpha\beta = \gamma$ then the above theorem gives :

$$(g\phi_{\alpha,\gamma})f = (gf)\phi_{\alpha f,\gamma f}.$$

Notation In the following chapters when we consider the endomorphisms of the semigroup S we shall use multiplicative notation. When we consider the elements of the d.g. seminear-ring $E(S)$ we use additive notation.

Now we are ready to obtain results on seminear-rings of endomorphisms of inverse semigroups which is our target in the remaining chapters.

Chapter 5

Seminear-ring of endomorphisms I

5.1 Starting case I

Let

$$Y = \{0, 1\}$$

be a semilattice with $0 \leq 1$.

Suppose that G is a group and let G_0 and G_1 be groups which are isomorphic to G, i.e.

$$G_1 \cong G \cong G_0.$$

Let

$$\phi_{1,0} : G_1 \longrightarrow G_0$$

be a homomorphism, which is indeed an isomorphism.

Consider $S = G_0 \bigcup G_1$, and let $f \in \mathrm{End}(S)$.

By theorem 4.2.3, $f|_Y$ is an endomorphism of Y which, in this case, will give the following endomorphisms:

$$\begin{array}{lll} f_1 : 1 \longrightarrow 1 & \qquad f_2 : 1 \longrightarrow 1 & \qquad f_3 : 1 \longrightarrow 0 \\ \quad\; 0 \longrightarrow 0 & \qquad\quad\; 0 \longrightarrow 1 & \qquad\quad\; 0 \longrightarrow 0 \end{array}$$

We shall refer to f_1, f_2 and f_3 as the endomorphisms of type I , type II and type III respectively.

By theorem 4.2.4,

$$(g\phi_{1,0})f = (gf)\phi_{1f,0f}.$$

So given which type f is, the action of f on G_0 is defined by the action of f on G_1.

If we consider the above endomorphisms of type I, then by theorem 4.2.3, for any endomorphism f of type I on S, we have

$$f|_{G_1} \in \mathrm{End}(G_1) \cong \mathrm{End}(G) \quad \text{and} \quad f|_{G_0} \in \mathrm{End}(G_0) \cong \mathrm{End}(G).$$

Thus, for each endomorphism f of type I on S, there would be an endomorphism on G. Again for an endomorphism f of type II on S, we have

$$f|_{G_1} \in \mathrm{End}(G_1) \cong \mathrm{End}(G) \text{ and } \quad f|_{G_0} \in \mathrm{Hom}(G_0, G_1) \cong \mathrm{End}(G).$$

We notice that the first one of the above isomorphisms is an isomorphism of semigroups while the second one is a 1-1 correspondence. The same holds for endomorphisms of type III. This implies that for each endomorphism f of a specific type on S, there is an endomorphism on the group G and the following correspondence holds

$$\{f|_{G_1}\, ;\, f \in \mathrm{End}(S)\} \longleftrightarrow \{f|_{G_0}\, ;\, f \in \mathrm{End}(S)\} \longleftrightarrow \mathrm{End}(G).$$

Now we may proceed the other way around and pick up an arbitrary endomorphism of G, α, say. Recall that $G_1 \cong G \cong G_0$, then we may consider the following isomorphisms:

$$\theta_1 : G \longrightarrow G_1\ ,$$
$$\theta_0 : G \longrightarrow G_0.$$

Let

$$\alpha^{(1)} : S \longrightarrow S$$

be a map defined via α as follows

$$(s)\alpha^{(1)} = \begin{cases} s\theta_1^{-1}\alpha\theta_1 & \text{if } s \in G_1\ , \\ s\theta_0^{-1}\alpha\theta_0 & \text{if } s \in G_0 \end{cases}$$

then $\alpha^{(1)}$ is an endomorphism of S.

To see that we first need to show that Theorem 4.2.4 is satisfied for $\alpha^{(1)}$, that is for all $g \in G_1$, we must have

$$(g)\phi_{1,0}\alpha^{(1)} = (g)\alpha^{(1)}\phi_{1,0}. \tag{5.1}$$

Since we already have the isomorphisms $\theta_1 : G \longrightarrow G_1$, and $\phi_{1,0} : G_1 \longrightarrow G_0$, and G is isomorphic to G_0 under θ_0, we can define θ_0 by $\theta_0 = \theta_1\phi_{1,0}$ and hence

$$\begin{aligned}
\theta_0^{-1} &= \phi_{1,0}^{-1}\theta_1^{-1} \\
\theta_0^{-1}\alpha\theta_0 &= \phi_{1,0}^{-1}\theta_1^{-1}\alpha\theta_1\phi_{1,0} \\
\phi_{1,0}\theta_0^{-1}\alpha\theta_0 &= \theta_1^{-1}\alpha\theta_1\phi_{1,0}.
\end{aligned}$$

So that for $g \in G_1$, we have

$$\begin{aligned}
(g)\phi_{1,0}\theta_0^{-1}\alpha\theta_0 &= (g)\theta_1^{-1}\alpha\theta_1\phi_{1,0} \\
(g)\phi_{1,0}\alpha^{(1)} &= (g)\alpha^{(1)}\phi_{1,0}, \text{ as required.}
\end{aligned}$$

Now we show that $\alpha^{(1)} \in \mathrm{End}(S)$.

Let $s_1, s_2 \in S$, then there are three possibilities :

Case i If $s_1, s_2 \in G_1$, then

$$\begin{aligned}
(s_1s_2)\alpha^{(1)} &= (s_1s_2)\theta_1^{-1}\alpha\theta_1 \\
&= ((s_1)\theta_1^{-1}\alpha\theta_1)((s_2)\theta_1^{-1}\alpha\theta_1) \\
&= (s_1)\alpha^{(1)}(s_2)\alpha^{(1)}.
\end{aligned}$$

Case ii If $s_1, s_2 \in G_0$, then

$$\begin{aligned}
(s_1s_2)\alpha^{(1)} &= (s_1s_2)\theta_0^{-1}\alpha\theta_0 \\
&= ((s_1)\theta_0^{-1}\alpha\theta_0)((s_2)\theta_0^{-1}\alpha\theta_0) \\
&= (s_1)\alpha^{(1)}(s_2)\alpha^{(1)}.
\end{aligned}$$

Case iii If $s_1 \in G_1, s_2 \in G_0$,(similarly if $s_1 \in G_0, s_2 \in G_1$), then

$$(s_1s_2)\alpha^{(1)} = (s_0s_2)\alpha^{(1)}, \text{where } s_0 = (s_1)\phi_{1,0} \in G_0$$

$$
\begin{aligned}
&= (s_0 s_2)\theta_0^{-1}\alpha\theta_0 \\
&= ((s_0)\theta_0^{-1}\alpha\theta_0)((s_2)\theta_0^{-1}\alpha\theta_0) \\
&= (s_0)\alpha^{(1)}(s_2)\alpha^{(1)} \\
&= ((s_1)\phi_{1,0})\alpha^{(1)}(s_2)\alpha^{(1)} \\
&= ((s_1)\alpha^{(1)})\phi_{1,0}(s_2)\alpha^{(1)}, \text{ by (5.1).} \\
&= (s_1)\alpha^{(1)}(s_2)\alpha^{(1)}.
\end{aligned}
$$

Hence $\alpha^{(1)}$ is an endomorphism of S.
From the definition of $\alpha^{(1)}$, we observe that

$$(1)\alpha^{(1)} = 1 \quad \text{and} \quad (0)\alpha^{(1)} = 0,$$

which means that $\alpha^{(1)}$ is an endomorphism of type I on S.

Next again we define a map $\alpha^{(2)}$ via α as

$$\alpha^{(2)} : S \longrightarrow S$$

$$(s)\alpha^{(2)} = \begin{cases} s\theta_1^{-1}\alpha\theta_1 & \text{if } s \in G_1 , \\ s\theta_0^{-1}\alpha\theta_1 & \text{if } s \in G_0 \end{cases}$$

then $\alpha^{(2)}$ is an endomorphism of S, as we now see.

Case i If $s_1, s_2 \in G_1$, then

$$
\begin{aligned}
(s_1 s_2)\alpha^{(2)} &= (s_1 s_2)\theta_1^{-1}\alpha\theta_1 \\
&= ((s_1)\theta_1^{-1}\alpha\theta_1)((s_2)\theta_1^{-1}\alpha\theta_1) \\
&= (s_1)\alpha^{(2)}(s_2)\alpha^{(2)}.
\end{aligned}
$$

Case ii If $s_1, s_2 \in G_0$, then

$$
\begin{aligned}
(s_1 s_2)\alpha^{(2)} &= (s_1 s_2)\theta_0^{-1}\alpha\theta_1 \\
&= ((s_1)\theta_0^{-1}\alpha\theta_1)((s_2)\theta_0^{-1}\alpha\theta_1) \\
&= (s_1)\alpha^{(2)}(s_2)\alpha^{(2)}.
\end{aligned}
$$

Case iii If $s_1 \in G_1, s_2 \in G_0$, (similarly if $s_1 \in G_0, s_2 \in G_1$), then in this case we first need to show that $\alpha^{(2)}$ must satisfy relation (5.2) for all $g \in G_1$:

$$(g)\phi_{1,0}\alpha^{(2)} = (g)\alpha^{(2)}. \tag{5.2}$$

Since we already have $\theta_0^{-1} = \phi_{1,0}^{-1}\theta_1^{-1}$, so

$$\begin{aligned} \theta_0^{-1}\alpha\theta_1 &= \phi_{1,0}^{-1}\theta_1^{-1}\alpha\theta_1 \\ \phi_{1,0}\theta_0^{-1}\alpha\theta_1 &= \theta_1^{-1}\alpha\theta_1 \end{aligned}$$

So that for $g \in G_1$, we have

$$\begin{aligned} (g)\phi_{1,0}\theta_0^{-1}\alpha\theta_1 &= (g)\theta_1^{-1}\alpha\theta_1 \\ (g)\phi_{1,0}\alpha^{(2)} &= (g)\alpha^{(2)}, \text{ and (5.2) is satisfied.} \end{aligned}$$

Coming back to case iii, we can see that

$$\begin{aligned} (s_1s_2)\alpha^{(2)} &= (s_0s_2)\alpha^{(2)}, \text{ where } s_0 = (s_1)\phi_{1,0} \in G_0 \\ &= (s_0s_2)\theta_0^{-1}\alpha\theta_1 \\ &= ((s_0)\theta_0^{-1}\alpha\theta_1)((s_2)\theta_0^{-1}\alpha\theta_1) \\ &= (s_0)\alpha^{(2)}(s_2)\alpha^{(2)} \\ &= ((s_1)\phi_{1,0})\alpha^{(2)}(s_2)\alpha^{(2)} \\ &= (s_1)\alpha^{(2)}(s_2)\alpha^{(2)}, \text{ by (5.2).} \end{aligned}$$

Hence $\alpha^{(2)}$ is an endomorphism of S.

From the definition of $\alpha^{(2)}$, we observe that

$$(1)\alpha^{(2)} = 1 \quad \text{and} \quad (0)\alpha^{(2)} = 1,$$

which implies that $\alpha^{(2)}$ is an endomorphism of type II on S.

Finally we define a map $\alpha^{(3)}$ via α as

$$\alpha^{(3)} : S \longrightarrow S$$

$$(s)\alpha^{(3)} = \begin{cases} s\theta_1^{-1}\alpha\theta_0 & \text{if } s \in G_1 , \\ s\theta_0^{-1}\alpha\theta_0 & \text{if } s \in G_0 \end{cases}$$

then $\alpha^{(3)}$ is an endomorphism of S, as we now see.

Case i If $s_1, s_2 \in G_1$, then

$$\begin{aligned} (s_1s_2)\alpha^{(3)} &= (s_1s_2)\theta_1^{-1}\alpha\theta_0 \\ &= ((s_1)\theta_1^{-1}\alpha\theta_0)((s_2)\theta_1^{-1}\alpha\theta_0) \\ &= (s_1)\alpha^{(3)}(s_2)\alpha^{(3)}. \end{aligned}$$

Case ii If $s_1, s_2 \in G_0$, then

$$\begin{aligned}(s_1s_2)\alpha^{(3)} &= (s_1s_2)\theta_0^{-1}\alpha\theta_0\\ &= ((s_1)\theta_0^{-1}\alpha\theta_0)((s_2)\theta_0^{-1}\alpha\theta_0)\\ &= (s_1)\alpha^{(3)}(s_2)\alpha^{(3)}.\end{aligned}$$

Case iii If $s_1 \in G_1, s_2 \in G_0$, (similarly if $s_1 \in G_0, s_2 \in G_1$), then in this case we have to show that $\alpha^{(2)}$ must satisfy relation (5.3) for all $g \in G_1$:

$$(g)\phi_{1,0}\alpha^{(3)} = (g)\alpha^{(3)}. \tag{5.3}$$

Since we already have $\theta_0^{-1} = \phi_{1,0}^{-1}\theta_1^{-1}$, so

$$\begin{aligned}\theta_0^{-1}\alpha\theta_0 &= \phi_{1,0}^{-1}\theta_1^{-1}\alpha\theta_0\\ \phi_{1,0}\theta_0^{-1}\alpha\theta_0 &= \theta_1^{-1}\alpha\theta_0.\end{aligned}$$

So that for $g \in G_1$, we have

$$\begin{aligned}(g)\phi_{1,0}\theta_0^{-1}\alpha\theta_0 &= (g)\theta_1^{-1}\alpha\theta_0\\ (g)\phi_{1,0}\alpha^{(3)} &= (g)\alpha^{(3)}, \text{ and (5.3) is satisfied.}\end{aligned}$$

Thus, we now have

$$\begin{aligned}(s_1s_2)\alpha^{(3)} &= (s_0s_2)\alpha^{(3)}, \text{where } s_0 = (s_1)\phi_{1,0} \in G_0\\ &= (s_0s_2)\theta_0^{-1}\alpha\theta_0\\ &= ((s_0)\theta_0^{-1}\alpha\theta_0)((s_2)\theta_0^{-1}\alpha\theta_0)\\ &= (s_0)\alpha^{(3)}(s_2)\alpha^{(3)}\\ &= ((s_1)\phi_{1,0})\alpha^{(3)}(s_2)\alpha^{(3)}\\ &= (s_1)\alpha^{(3)}(s_2)\alpha^{(3)}, \text{ by (5.3).}\end{aligned}$$

Hence $\alpha^{(3)}$ is an endomorphism of S.

It can be seen that

$$(1)\alpha^{(3)} = 0 \quad \text{and} \quad (0)\alpha^{(3)} = 0,$$

which means that $\alpha^{(3)}$ is an endomorphism of type III on S.

From above, we observe that an arbitrary endomorphism of G could give rise to an endomorphism of the semilattice S of each type.

Let us now link up $\mathrm{End}(G)$ with $\mathrm{End}(S)$ by defining a map Γ_1 where,

$$\Gamma_1 : \mathrm{End}(G) \longrightarrow \mathrm{End}(S)$$

is given by

$$(\alpha)\Gamma_1 = \alpha^{(1)}.$$

The map Γ_1 is a homomorphism for, if $\alpha,\ \beta \in \mathrm{End}(G)$, then for $s \in G_1$, we have

$$\begin{aligned} s(\alpha\beta)\Gamma_1 &= s(\gamma)\Gamma_1, \text{ where } \gamma = \alpha\beta, \\ &= (s)\gamma^{(1)} \\ &= (s)\theta_1^{-1}\gamma\theta_1 \\ &= (s)\theta_1^{-1}\alpha\beta\theta_1 \\ &= (s)\theta_1^{-1}\alpha\theta_1\theta_1^{-1}\beta\theta_1 \\ &= (s)(\theta_1^{-1}\alpha\theta_1)(\theta_1^{-1}\beta\theta_1) \\ &= (s)\alpha^{(1)}\beta^{(1)} \\ &= (s)(\alpha\Gamma_1)(\beta\Gamma_1). \end{aligned}$$

Similarly when $s \in G_0$, we can show that

$$s(\alpha\beta)\Gamma_1 = (s)(\alpha\Gamma_1)(\beta\Gamma_1).$$

Hence

$$(\alpha\beta)\Gamma_1 = (\alpha\Gamma_1)(\beta\Gamma_1) \text{ and } \Gamma_1 \text{ is a homomorphism.}$$

Moreover, Γ_1 is a monomorphism for if $\alpha \neq \beta$ in $\mathrm{End}(G)$, then there exists $g \in G$ such that $g\alpha \neq g\beta$. But $g = g_1\theta_1^{-1}$ for some $g_1 \in G_1$, which gives

$$\begin{aligned} g_1\theta_1^{-1}\alpha &\neq g_1\theta_1^{-1}\beta, \\ g_1\theta_1^{-1}\alpha\theta_1 &\neq g_1\theta_1^{-1}\beta\theta_1, \text{as } \theta_1 \text{ is an isomorphism,} \\ g_1\alpha^{(1)} &\neq g_1\beta^{(1)}, \\ (\alpha)\Gamma_1 &\neq (\beta)\Gamma_1. \end{aligned}$$

Hence, Γ_1 is a monomorphism.

Next we define a map Γ_2 where,

$$\Gamma_2 : \mathrm{End}(G) \longrightarrow \mathrm{End}(S)$$

is given by

$$(\alpha)\Gamma_2 = \alpha^{(2)}.$$

We show that the map Γ_2 is a homomorphism. Let $\alpha,\ \beta \in \mathrm{End}(G)$, then for $s \in G_1$, we have

$$\begin{aligned}
s(\alpha\beta)\Gamma_2 &= s(\gamma)\Gamma_2, \text{ where } \gamma = \alpha\beta, \\
&= (s)\gamma^{(2)} \\
&= (s)\theta_1^{-1}\gamma\theta_1 \\
&= (s)\theta_1^{-1}\alpha\beta\theta_1 \\
&= (s)\theta_1^{-1}\alpha\theta_1\theta_1^{-1}\beta\theta_1 \\
&= (s)(\theta_1^{-1}\alpha\theta_1)(\theta_1^{-1}\beta\theta_1) \\
&= (s)\alpha^{(2)}\beta^{(2)} \\
&= (s)(\alpha\Gamma_2)(\beta\Gamma_2),
\end{aligned}$$

and for $s \in G_0$, we have

$$\begin{aligned}
s(\alpha\beta)\Gamma_2 &= s(\gamma)\Gamma_2, \text{ where } \gamma = \alpha\beta, \\
&= (s)\gamma^{(2)} \\
&= (s)\theta_0^{-1}\gamma\theta_1 \\
&= (s)\theta_0^{-1}\alpha\beta\theta_1 \\
&= (s)\theta_0^{-1}\alpha\theta_1\theta_1^{-1}\beta\theta_1 \\
&= (s)(\theta_0^{-1}\alpha\theta_1)(\theta_1^{-1}\beta\theta_1) \\
&= (s)\alpha^{(2)}\beta^{(2)} \\
&= (s)(\alpha\Gamma_2)(\beta\Gamma_2).
\end{aligned}$$

Hence

$$(\alpha\beta)\Gamma_2 \;=\; (\alpha\Gamma_2)(\beta\Gamma_2) \text{ and } \Gamma_2 \text{ is a homomorphism.}$$

Furthermore, by a similar way applied to Γ_1, we show that Γ_2 is a monomorphism. The same argument can be applied to prove the existence of the monomorphism Γ_3, where

$$\Gamma_3 : \mathrm{End}(G) \longrightarrow \mathrm{End}(S)$$

is given by

$$(\alpha)\Gamma_3 = \alpha^{(3)}.$$

Now we may extend the maps $\Gamma_j, j = 1,2,3$, to be defined from the near-ring $E(G)$ to the seminear-ring $E(S)$ by defining the maps $\Gamma_j^\star, j = 1,2,3$, respectively, where :

$$\Gamma_j^\star, j = 1,2,3, : E(G) \longrightarrow E(S)$$

are given by

$$(\sum \epsilon_i \alpha_i)\Gamma_j^\star = \sum \epsilon_i \alpha_i^{(j)}.$$

where $\epsilon_i = \pm 1$.

We show that the maps $\Gamma_j^\star, j = 1,2,3$, are homomorphisms of groups.

Consider first $\Gamma_1^\star$, then to show it is a homomorphism, we only need to show that

$$(c)\Gamma_1^\star = 0 \text{ whenever } 0 = c \in E(G).$$

Suppose that $c \in E(G)$ such that $c = \sum_{i=1}^n \epsilon_i \alpha_i = 0$, then for any $g \in G$, we have

$$g \sum_{i=1}^n \epsilon_i \alpha_i = 0.$$

Considering $g = g_0\theta_0^{-1}$ for some $g_0 \in G_0$, we get

$$\begin{aligned} g_0\theta_0^{-1} \sum_{i=1}^n \epsilon_i \alpha_i \;&=\; 0 \\ g_0 \sum_{i=1}^n \epsilon_i \theta_0^{-1} \alpha_i \;&=\; 0 \end{aligned}$$

$$
\begin{aligned}
g_0(\sum_{i=1}^{n}\epsilon_i\theta_0^{-1}\alpha_i)\theta_0 &= 0\\
g_0(\sum_{i=1}^{n}\epsilon_i\theta_0^{-1}\alpha_i\theta_0) &= 0\\
g_0\sum_{i=1}^{n}\epsilon_i\alpha_i^{(1)} &= 0\\
g_0(\sum_{i=1}^{n}\epsilon_i\alpha_i)\Gamma_1^\star &= 0\\
g_0(c)\Gamma_1^\star &= 0.
\end{aligned}
$$

Considering again $g = g_1\theta_1^{-1}$ for some $g_1 \in G_1$, we get

$$
\begin{aligned}
g_1\theta_1^{-1}\sum_{i=1}^{n}\epsilon_i\alpha_i &= 0\\
g_1\sum_{i=1}^{n}\epsilon_i\theta_1^{-1}\alpha_i &= 0\\
g_1(\sum_{i=1}^{n}\epsilon_i\theta_1^{-1}\alpha_i)\theta_1 &= 0\\
g_1(\sum_{i=1}^{n}\epsilon_i\theta_1^{-1}\alpha_i\theta_1) &= 0\\
g_1\sum_{i=1}^{n}\epsilon_i\alpha_i^{(1)} &= 0\\
g_1(\sum_{i=1}^{n}\epsilon_i\alpha_i)\Gamma_1^\star &= 0\\
g_1(c)\Gamma_1^\star &= 0.
\end{aligned}
$$

So

$$g(c)\Gamma_1^\star = 0 \text{ for all } g \in S,$$

which means that

$$(c)\Gamma_1^\star = 0.$$

Hence $\Gamma_1^\star$ is a homomorphism. Similarly we can show that $\Gamma_2^\star$ and $\Gamma_3^\star$ are homomorphisms.

Finally we show that $\Gamma_j^\star, j = 1,2,3$, are monomorphisms.

Consider $\Gamma_1^\star$ and suppose that $0 \neq c \in E(G)$. We show that $(c)\Gamma_1^\star \neq 0$.

Let $c = \sum_{i=1}^{r}\epsilon_i\alpha_i$ and suppose that $(g)\sum_{i=1}^{r}\epsilon_i\alpha_i \neq 0$ for some $g \in G$.

For this element g, there exists an element $s \in G_1$ such that $s = g\theta_1$. Thus

$$\begin{aligned}
s(c)\Gamma_1^\star &= (s)(\sum_{i=1}^{r} \epsilon_i \alpha_i)\Gamma_1^\star \\
&= (s)(\sum_{i=1}^{r} \epsilon_i \alpha_i^{(1)}) \\
&= g\theta_1(\sum_{i=1}^{r} \epsilon_i \alpha_i^{(1)}) \\
&= g(\sum_{i=1}^{r} \epsilon_i \theta_1 \alpha_i^{(1)}) \\
&= g(\sum_{i=1}^{r} \epsilon_i \theta_1 \theta_1^{-1} \alpha_i \theta_1) \\
&= g(\sum_{i=1}^{r} \epsilon_i \alpha_i \theta_1) \\
&= g(\sum_{i=1}^{r} \epsilon_i \alpha_i)\theta_1 \\
&\neq 0 \text{ , since } \theta_1 \text{ is one to one.}
\end{aligned}$$

This shows that $\Gamma_1^\star$ is a monomorphism. Similarly one can show that $\Gamma_2^\star$ and $\Gamma_3^\star$ are monomorphisms. By lemma 2.2.5, it follows that $\Gamma_j^\star, j = 1, 2, 3$, are indeed d.g. homomorphisms from $E(G)$ into $E(S)$. Hence, as End(G) generates the d.g. near-ring $E(G)$, we deduce that the endomorphisms of type I on S will generate in $E(S)$ a subnear-ring isomorphic to the near-ring $E(G)$. Similarly the endomorphisms of type II on S will generate in $E(S)$ a subnear-ring isomorphic to $E(G)$. The same holds for the endomorphisms of type III.

5.2 Addition in $E(S)$

So far we have got three copies of the near-ring $E(G)$ inside the seminear-ring $E(S)$, which have been formed by the endomorphisms of those types as described. The question could be raised now : what can be said about the sum of endomorphisms of different types on S ? The answer to this question will give the ability to determine the sum inside $E(S)$. So let us start by studying the sum of endomorphisms of type I with type II.

Suppose that $\alpha^{(1)}$ and $\gamma^{(2)}$ are two endomorphisms of S of type I and type II respectively. Then for $s \in G_1$ we have

$$\begin{aligned} s(\alpha^{(1)} + \gamma^{(2)}) &= s\alpha^{(1)} + s\gamma^{(2)} \\ &= s\alpha^{(1)} + s(\theta_1^{-1}\gamma\theta_1) \\ &= s\alpha^{(1)} + s\gamma^{(1)} \\ &= s(\alpha^{(1)} + \gamma^{(1)}), \end{aligned} \tag{5.4}$$

and for $s \in G_0$, we have

$$\begin{aligned} s(\alpha^{(1)} + \gamma^{(2)}) &= s\alpha^{(1)} + s\gamma^{(2)} \\ &= s\alpha^{(1)} + s(\theta_0^{-1}\gamma\theta_1) \\ &= s\alpha^{(1)} + s(\theta_0^{-1}\gamma\theta_1)\phi_{1,0} \\ &= s\alpha^{(1)} + s\theta_0^{-1}\gamma\theta_0 \\ &= s\alpha^{(1)} + s\gamma^{(1)} \\ &= s(\alpha^{(1)} + \gamma^{(1)}). \end{aligned} \tag{5.5}$$

Equations (5.4) and (5.5) imply that

$$\alpha^{(1)} + \gamma^{(2)} = \alpha^{(1)} + \gamma^{(1)}. \tag{5.6}$$

Although addition here is not necessarily commutative, the same conclusion can be obtained if we reverse the order of addition of the above maps. Thus we get

$$\gamma^{(2)} + \alpha^{(1)} = \gamma^{(1)} + \alpha^{(1)}. \tag{5.7}$$

Equations(5.6) and (5.7) show that the sum of two endomorphisms of type I and type II on S could be assumed to be a sum of endomorphisms of type I.

Next consider endomorphisms $\alpha^{(1)}$ and $\beta^{(3)}$ of type I and type III respectively. For $s \in G_1$, we have

$$\begin{aligned} s(\alpha^{(1)} + \beta^{(3)}) &= s\alpha^{(1)} + s\beta^{(3)} \\ &= s\theta_1^{-1}\alpha\theta_1 + s\beta^{(3)} \\ &= (s\theta_1^{-1}\alpha\theta_1)\phi_{1,0} + s\beta^{(3)} \end{aligned}$$

$$
\begin{aligned}
&= s\theta_1^{-1}\alpha\theta_0 + s\beta^{(3)} \\
&= s\alpha^{(3)} + s\beta^{(3)} \\
&= s(\alpha^{(3)} + \beta^{(3)}), \qquad (5.8)
\end{aligned}
$$

and for $s \in G_0$, we have

$$
\begin{aligned}
s(\alpha^{(1)} + \beta^{(3)}) &= s\alpha^{(1)} + s\beta^{(3)} \\
&= s\theta_0^{-1}\alpha\theta_0 + s\beta^{(3)} \\
&= s\alpha^{(3)} + s\beta^{(3)} \\
&= s(\alpha^{(3)} + \beta^{(3)}). \qquad (5.9)
\end{aligned}
$$

Equations (5.8) and (5.9) imply that

$$
\alpha^{(1)} + \beta^{(3)} = \alpha^{(3)} + \beta^{(3)}. \qquad (5.10)
$$

Similarly we can get

$$
\beta^{(3)} + \alpha^{(1)} = \beta^{(3)} + \alpha^{(3)}. \qquad (5.11)
$$

Equations (5.10) and (5.11) show that the sum of two endomorphisms of type I and type III on S could be regarded as a sum of endomorphisms of type III.
Finally we consider the sum of $\gamma^{(2)}$ with $\beta^{(3)}$; then for $s \in G_1$, we have

$$
\begin{aligned}
s(\gamma^{(2)} + \beta^{(3)}) &= s\gamma^{(2)} + s\beta^{(3)} \\
&= s\theta_1^{-1}\gamma\theta_1 + s\beta^{(3)} \\
&= (s\theta_1^{-1}\gamma\theta_1)\phi_{1,0} + s\beta^{(3)} \\
&= s\theta_1^{-1}\gamma\theta_0 + s\beta^{(3)} \\
&= s\gamma^{(3)} + s\beta^{(3)} \\
&= s(\gamma^{(3)} + \beta^{(3)}), \qquad (5.12)
\end{aligned}
$$

and for $s \in G_0$, we have

$$
s(\gamma^{(2)} + \beta^{(3)}) = s\gamma^{(2)} + s\beta^{(3)}
$$

$$
\begin{aligned}
&= s\theta_0^{-1}\gamma\theta_1 + s\beta^{(3)} \\
&= (s\theta_0^{-1}\gamma\theta_1)\phi_{1,0} + s\beta^{(3)} \\
&= s\theta_0^{-1}\gamma\theta_0 + s\beta^{(3)} \\
&= s\gamma^{(3)} + s\beta^{(3)} \\
&= s(\gamma^{(3)} + \beta^{(3)}). \qquad (5.13)
\end{aligned}
$$

Equations (5.12) and (5.13) imply that

$$\gamma^{(2)} + \beta^{(3)} = \gamma^{(3)} + \beta^{(3)}. \qquad (5.14)$$

Similarly we can get

$$\beta^{(3)} + \gamma^{(2)} = \beta^{(3)} + \gamma^{(3)}. \qquad (5.15)$$

Equations (5.14) and (5.15) show that the sum of two endomorphisms of type II and type III on S could be regarded as a sum of endomorphisms of type III. In order to continue in determining the sum inside $E(S)$, we have to consider the d.g. near-rings generated by the endomorphisms of type I, type II and type III which we will denote by $E(G)^I, E(G)^{II}$ and $E(G)^{III}$, respectively. Thus from previous discussion we have

$$E(S) = E(G)^I \cup E(G)^{II} \cup E(G)^{III}. \qquad (5.16)$$

Using induction we can show that the sum of a finite number of endomorphisms of type I on S with a finite number of endomorphisms of type II on S could be considered as sum of endomorphisms of type I on S. In other words, the sum of an element in $E(G)^I$ with an element in $E(G)^{II}$ could be assumed as a sum occuring in $E(G)^I$. To see that we have to prove that the following equation is satisfied :

$$\sum_{i=1}^{n} \epsilon_i \alpha_i^{(1)} + \sum_{j=1}^{k} \eta_j \gamma_j^{(2)} = \sum_{i=1}^{n} \epsilon_i \alpha_i^{(1)} + \sum_{j=1}^{k} \eta_j \gamma_j^{(1)} \qquad (5.17)$$

where,

$\alpha_i^{(1)}$, $\gamma_j^{(1)} \in$ type I , $\gamma_j^{(2)} \in$ type II, and $\epsilon_i = \pm 1$, $\eta_j = \pm 1$, $i = 1, \ldots, n$, $j = 1, \ldots, k$.

We apply induction on k in equation (5.17).

From above, it is evident that equation (5.17) is valid for $k = 1$ since we already have equation (5.6) which will easily give

$$\sum_{i=1}^{n} \epsilon_i \alpha_i^{(1)} + \eta_1 \gamma_1^{(2)} = \sum_{i=1}^{n} \epsilon_i \alpha_i^{(1)} + \eta_1 \gamma_1^{(1)}. \tag{5.18}$$

Suppose that equation (5.17) is true for $k-1$, then

$$\begin{aligned}
\sum_{i=1}^{n} \epsilon_i \alpha_i^{(1)} + \sum_{j=1}^{k} \eta_j \gamma_j^{(2)} &= \sum_{i=1}^{n} \epsilon_i \alpha_i^{(1)} + \sum_{j=1}^{k-1} \eta_j \gamma_j^{(2)} + \eta_k \gamma_k^{(2)} \\
&= \sum_{i=1}^{n} \epsilon_i \alpha_i^{(1)} + \sum_{j=1}^{k-1} \eta_j \gamma_j^{(1)} + \eta_k \gamma_k^{(2)}, \text{ by hypothesis,} \\
&= \sum_{i=1}^{n} \epsilon_i \alpha_i^{(1)} + \sum_{j=1}^{k-1} \eta_j \gamma_j^{(1)} + \eta_k \gamma_k^{(1)} \text{ , by equation (5.18),} \\
&= \sum_{i=1}^{n} \epsilon_i \alpha_i^{(1)} + \sum_{j=1}^{k} \eta_j \gamma_j^{(1)}.
\end{aligned}$$

Hence equation (5.17) is satisfied and the sum of an element in $E(G)^I$ with an element in $E(G)^{II}$ lies in $E(G)^I$.

Similar induction arguments could be applied to prove each of equations (5.19) and (5.20), where

$$\sum_{i=1}^{n} \epsilon_i \alpha_i^{(1)} + \sum_{r=1}^{m} \delta_r \beta_r^{(3)} = \sum_{i=1}^{n} \epsilon_i \alpha_i^{(3)} + \sum_{r=1}^{m} \delta_r \beta_r^{(3)} \tag{5.19}$$

$$\sum_{j=1}^{k} \eta_j \gamma_j^{(2)} + \sum_{r=1}^{m} \delta_r \beta_r^{(3)} = \sum_{j=1}^{k} \eta_j \gamma_j^{(3)} + \sum_{r=1}^{m} \delta_r \beta_r^{(3)} \tag{5.20}$$

where,

$\alpha_i^{(1)} \in$ type I , $\gamma_j^{(2)} \in$ type II , $\beta_r^{(3)}, \alpha_i^{(3)}, \gamma_j^{(3)} \in$ type III and $\epsilon_i = \pm 1$, $\delta_r = \pm 1$, $\eta_j = \pm 1$, $i = 1, \ldots, n$, $r = 1, \ldots, m$, $j = 1, \ldots, k$.

Equation (5.19) shows that the sum of an element in $E(G)^I$ with an element in $E(G)^{III}$ lies in $E(G)^{III}$, and equation (5.20) shows that the sum of an element in $E(G)^{II}$ with an element in $E(G)^{III}$ lies in $E(G)^{III}$. As mentioned earlier, the same conclusion can be obtained when changing the order of the above additions of those maps. So we have the following picture :

$$E(G)^I + E(G)^{II} \longrightarrow E(G)^I \quad \text{and} \quad E(G)^{II} + E(G)^I \longrightarrow E(G)^I$$

$$E(G)^I + E(G)^{III} \longrightarrow E(G)^{III} \quad \text{and} \quad E(G)^{III} + E(G)^I \longrightarrow E(G)^{III}$$

$$E(G)^{II} + E(G)^{III} \longrightarrow E(G)^{III} \quad \text{and} \quad E(G)^{III} + E(G)^{II} \longrightarrow E(G)^{III}$$

Now we turn to find the product in $E(S)$.

5.3 Product in $E(S)$

We start by considering endomorphisms $\alpha^{(1)}$ and $\gamma^{(2)}$ of type I and type II on S, respectively. For $s \in G_1$, we have

$$\begin{aligned} s(\alpha^{(1)}\gamma^{(2)}) &= s(\theta_1^{-1}\alpha\theta_1)(\theta_1^{-1}\gamma\theta_1) \\ &= s(\theta_1^{-1}\alpha\gamma\theta_1) \\ &= s(\alpha\gamma)^{(2)}, \end{aligned}$$

and for $s \in G_0$, we have

$$\begin{aligned} s(\alpha^{(1)}\gamma^{(2)}) &= s(\theta_0^{-1}\alpha\theta_0)(\theta_0^{-1}\gamma\theta_1) \\ &= s(\theta_0^{-1}\alpha\gamma\theta_1) \\ &= s(\alpha\gamma)^{(2)}. \end{aligned}$$

So

$$\alpha^{(1)}\gamma^{(2)} = (\alpha\gamma)^{(2)}. \tag{5.21}$$

Now we change the order of the maps in the above product to get for $s \in G_1$,

$$\begin{aligned} s(\gamma^{(2)}\alpha^{(1)}) &= s(\theta_1^{-1}\gamma\theta_1)(\theta_1^{-1}\alpha\theta_1) \\ &= s(\theta_1^{-1}\gamma\alpha\theta_1) \\ &= s(\gamma\alpha)^{(2)}, \end{aligned}$$

and for $s \in G_0$, we have

$$s(\gamma^{(2)}\alpha^{(1)}) = s(\theta_0^{-1}\gamma\theta_1)(\theta_1^{-1}\alpha\theta_1)$$

$$
\begin{aligned}
&= s(\theta_0^{-1}\gamma\alpha\theta_1) \\
&= s(\gamma\alpha)^{(2)}.
\end{aligned}
$$

Thus

$$
\gamma^{(2)}\alpha^{(1)} = (\gamma\alpha)^{(2)}. \tag{5.22}
$$

Equations (5.21) and (5.22) show that the product of $\alpha^{(1)}$ with $\gamma^{(2)}$ could be considered as a product of endomorphisms of type II.

Next we consider the product of $\alpha^{(1)}$ and $\beta^{(3)}$ of endomorphisms of type I and type III respectively. For $s \in G_1$, we have

$$
\begin{aligned}
s(\alpha^{(1)}\beta^{(3)}) &= s(\theta_1^{-1}\alpha\theta_1)(\theta_1^{-1}\beta\theta_0) \\
&= s(\theta_1^{-1}\alpha\beta\theta_0) \\
&= s(\alpha\beta)^{(3)},
\end{aligned}
$$

and for $s \in G_0$, we have

$$
\begin{aligned}
s(\alpha^{(1)}\beta^{(3)}) &= s(\theta_0^{-1}\alpha\theta_0)(\theta_0^{-1}\beta\theta_0) \\
&= s(\theta_0^{-1}\alpha\beta\theta_0) \\
&= s(\alpha\beta)^{(3)}.
\end{aligned}
$$

So

$$
\alpha^{(1)}\beta^{(3)} = (\alpha\beta)^{(3)}. \tag{5.23}
$$

Now we reverse the order of the maps in the above product to get for $s \in G_1$,

$$
\begin{aligned}
s(\beta^{(3)}\alpha^{(1)}) &= s(\theta_1^{-1}\beta\theta_0)(\theta_0^{-1}\alpha\theta_0) \\
&= s(\theta_1^{-1}\beta\alpha\theta_0) \\
&= s(\beta\alpha)^{(3)},
\end{aligned}
$$

and for $s \in G_0$, we have

$$
s(\beta^{(3)}\alpha^{(1)}) = s(\theta_0^{-1}\beta\theta_0)(\theta_0^{-1}\alpha\theta_0)
$$

$$\begin{aligned} &= s(\theta_0^{-1}\beta\alpha\theta_0) \\ &= s(\beta\alpha)^{(3)}. \end{aligned}$$

Thus

$$\beta^{(3)}\alpha^{(1)} = (\beta\alpha)^{(3)}. \tag{5.24}$$

Equations (5.23) and (5.24) show that the product of $\alpha^{(1)}$ with $\beta^{(3)}$ could be regarded as a product of endomorphisms of type III.

Finally we consider the product of $\gamma^{(2)}$ and $\beta^{(3)}$ of endomorphisms of type II and type III respectively. For $s \in G_1$, we have

$$\begin{aligned} s(\gamma^{(2)}\beta^{(3)}) &= s(\theta_1^{-1}\gamma\theta_1)(\theta_1^{-1}\beta\theta_0) \\ &= s(\theta_1^{-1}\gamma\beta\theta_0) \\ &= s(\gamma\beta)^{(3)}, \end{aligned}$$

and for $s \in G_0$, we have

$$\begin{aligned} s(\gamma^{(2)}\beta^{(3)}) &= s(\theta_0^{-1}\gamma\theta_1)(\theta_1^{-1}\beta\theta_0) \\ &= s(\theta_0^{-1}\gamma\beta\theta_0) \\ &= s(\gamma\beta)^{(3)} \end{aligned}$$

which shows that the product $\gamma^{(2)}\beta^{(3)}$ could be considered as a product of type III. However, the product $\beta^{(3)}\gamma^{(2)}$ does not behave as one might expect, as we can see that for $s \in G_1$, we have

$$\begin{aligned} s(\beta^{(3)}\gamma^{(2)}) &= s(\theta_1^{-1}\beta\theta_0)(\theta_0^{-1}\gamma\theta_1) \\ &= s(\theta_1^{-1}\beta\gamma\theta_1) \\ &= s(\beta\gamma)^{(2)} \end{aligned}$$

and for $s \in G_0$, we have

$$\begin{aligned} s(\beta^{(3)}\gamma^{(2)}) &= s(\theta_0^{-1}\beta\theta_0)(\theta_0^{-1}\gamma\theta_1) \\ &= s(\theta_0^{-1}\beta\gamma\theta_1) \\ &= s(\beta\gamma)^{(2)}. \end{aligned}$$

Thus the product $\beta^{(3)}\gamma^{(2)}$ could be considered as a product of endomorphisms of type II, while the product $\gamma^{(2)}\beta^{(3)}$ is considered as a product of endomorphisms of type III. So we have the following picture :

$$E(G)^{I}.E(G)^{II} \longrightarrow E(G)^{II} \quad \text{and} \quad E(G)^{II}.E(G)^{I} \longrightarrow E(G)^{II}$$

$$E(G)^{I}.E(G)^{III} \longrightarrow E(G)^{III} \quad \text{and} \quad E(G)^{III}.E(G)^{I} \longrightarrow E(G)^{III}$$

$$E(G)^{II}.E(G)^{III} \longrightarrow E(G)^{III} \quad \text{but} \quad E(G)^{III}.E(G)^{II} \longrightarrow E(G)^{II}.$$

Thus the product fails to construct a semilattice of groups.

Our final section will now give us the conclusion about the structure of $E(S)$.

5.4 Conclusion I

Let us return to addition in $E(S)$ and define the homomorphisms $\phi_{II,I}, \phi_{II,III}$ and $\phi_{I,III}$ while considering $III \leq I \leq II$, such that

$$\begin{aligned} \phi_{II,I} : E(G)^{II} &\longrightarrow E(G)^{I} \\ \textstyle\sum_j \eta_j \gamma_j^{(2)} &\longrightarrow \textstyle\sum_j \eta_j \gamma_j^{(1)}, \end{aligned}$$

$$\begin{aligned} \phi_{II,III} : E(G)^{II} &\longrightarrow E(G)^{III} \\ \textstyle\sum_j \eta_j \gamma_j^{(2)} &\longrightarrow \textstyle\sum_j \eta_j \gamma_j^{(3)}, \end{aligned}$$

$$\begin{aligned} \phi_{I,III} : E(G)^{I} &\longrightarrow E(G)^{III} \\ \textstyle\sum_i \epsilon_i \alpha_i^{(1)} &\longrightarrow \textstyle\sum_i \epsilon_i \alpha_i^{(3)}. \end{aligned}$$

Let $\mathcal{L} = \{I, II, III\}$, then we have a strong semilattice of additive groups given by

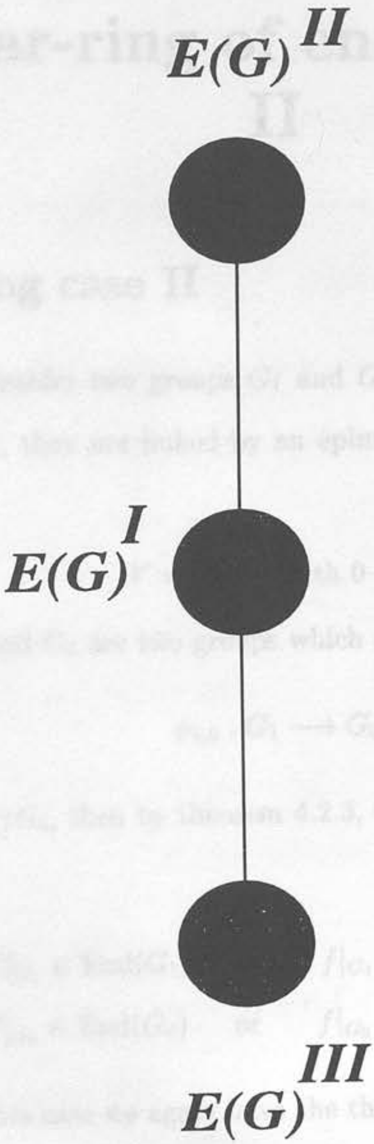
$$E(S) = (\mathcal{L}\ , \{E(G)^i\}_{i \in \mathcal{L}}\ , \{\phi_{II,I}, \phi_{II,III}, \phi_{I,III}\})$$

that is,

$(E(S), +)$ is a Clifford semigroup.

Chapter 6

Seminear-rings of endomorphisms II



[illegible] Starting case II

[illegible] we consider two groups G_1 and G_0 which are not necessarily isomorphic. However, they are linked by an epimorphism. So let us start with a [illegible]

[illegible] with $0 \leq 1$.

[illegible] that G_1 and G_0 are two groups which are linked by an epimorphism

$$\phi_{1,0} : G_1 \longrightarrow G_0.$$

[illegible], then by Theorem 4.2.3, for an endomorphism f on S, we have

$$f|_{G_1} \in \mathrm{End}(G_1) \quad \text{or} \quad f|_{G_1} \in \mathrm{Hom}(G_1, G_0)$$
$$f|_{G_0} \in \mathrm{End}(G_0) \quad \text{or} \quad f|_{G_0} \in \mathrm{Hom}(G_0, G_1).$$

[illegible] the three types of endomorphisms as in [illegible]. [illegible] that type I here will involve some relations which we will [illegible] will be discussing type I in detail. So let us suppose that f [illegible] of type I on S. Let $f|_{G_1} = \alpha$, say, where $\alpha \in \mathrm{End}(G_1)$ and [illegible], where $\beta \in \mathrm{End}(G_0)$. Then for $g_0 \in G_0$, we have

$$[illegible], \text{ for some } g_1 \in G_1 \text{ such that } g_1\phi_{1,0} = g_0.$$

Chapter 6

Seminear-ring of endomorphisms II

6.1 Starting case II

In this case we consider two groups G_1 and G_0 which are not necessarily isomorphic. However, they are linked by an epimorphism. So let us start with a semilattice

$$Y = \{0, 1\}, \text{ with } 0 \leq 1.$$

Suppose that G_1 and G_0 are two groups which are linked by an epimorphism

$$\phi_{1,0} : G_1 \longrightarrow G_0.$$

Consider $S = G_1 \cup G_0$, then by theorem 4.2.3, for an endomorphism f on S, we have

$$\begin{array}{lcl} f|_{G_1} \in \mathrm{End}(G_1) & \text{or} & f|_{G_1} \in \mathrm{Hom}(G_1, G_0) \\ f|_{G_0} \in \mathrm{End}(G_0) & \text{or} & f|_{G_0} \in \mathrm{Hom}(G_0, G_1). \end{array}$$

We note that in this case we again have the three types of endomorphisms as in chapter 5. Also we note that type I here will involve some relations which we will obtain, therefore we will be discussing type I in detail. So let us suppose that f is an endomorphism of type I on S. Let $f|_{G_1} = \alpha$, say, where $\alpha \in \mathrm{End}(G_1)$ and $f|_{G_0} = \beta$, say, where $\beta \in \mathrm{End}(G_0)$. Then for $g_0 \in G_0$, we have

$$g_0 f = (g_1\phi_{1,0})f \text{, for some } g_1 \in G_1 \text{ such that } g_1\phi_{1,0} = g_0,$$

$$\begin{aligned} &= (g_1 f)\phi_{1f,0f}\ , \text{ by theorem 4.2.4 },\\ &= (g_1 f)\phi_{1,0}\\ &= (g_1\alpha)\phi_{1,0}. \end{aligned} \tag{6.1}$$

Let e_i be the identity of $G_i, i = 0, 1$. Then for $g_1 \in G_1$, we have

$$\begin{aligned} (g_1 e_0)f &= (g_1\phi_{1,0}e_0)f\\ &= (g_1\phi_{1,0})f. \end{aligned} \tag{6.2}$$

Put Ker $\phi_{1,0} = \mathrm{K}$. Then for any $g \in \mathrm{K}$, we have

$$\begin{aligned} g\phi_{1,0} &= e_0\\ (g\phi_{1,0})f &= e_i \text{ ,for some } i \in \{0,\ 1\},\\ (gf)\phi_{1,0} &= e_0\ , \text{ by theorem 4.2.4 }. \end{aligned}$$

This shows that

$$\mathrm{K}f \subseteq \mathrm{K}$$

and hence for $k \in \mathrm{K}$, $g_1 \in G_1$, we have

$$\begin{aligned} (g_1 + k)f &= g_1 f + kf\\ &= g_1 f + k', \text{ for some } k' \in \mathrm{K}. \end{aligned}$$

That is

$$(g_1 + \mathrm{K})f \subseteq g_1 f + \mathrm{K}.$$

Since $f|_{G_1} = \alpha$, it follows that

$$\mathrm{K}\alpha \subseteq \mathrm{K}$$

and equation (6.2) gives

$$(g_1 e_0)f = (g_1\phi_{1,0})\beta. \tag{6.3}$$

On the other hand, we have

$$\begin{aligned}(g_1e_0)f &= (g_1f)(e_0f) \\ &= (g_1f)e_0 \\ &= (g_1\alpha)e_0 \\ &= (g_1\alpha)\phi_{1,0}e_0 \\ &= (g_1\alpha)\phi_{1,0}. \end{aligned} \tag{6.4}$$

Equations (6.3) and (6.4) imply that

$$\alpha\phi_{1,0} \;=\; \phi_{1,0}\beta. \tag{6.5}$$

If $g_1 \in \mathrm{K}$, the kernel of $\phi_{1,0}$, then by (6.5), we have

$$(g_1\alpha)\phi_{1,0} = (g_1\phi_{1,0})\beta = e_0\beta = e_0$$

and from above, we have

$$(g_1 + k)\alpha = g_1\alpha + k',\ k, k' \in \mathrm{K}.$$

So we can define a map

$$\bar{\alpha} : G_1/\mathrm{K} \longrightarrow G_1/\mathrm{K} \text{ by}$$

$$(g_1{+}\mathrm{K})\bar{\alpha} = g_1\alpha + \mathrm{K}.$$

First we show that this map is well-defined.

Let $g_1{+}\mathrm{K} = g_2{+}\ \mathrm{K}$ in G_1/K, then

$$\begin{aligned}(g_1 + \mathrm{K})\alpha &= (g_2 + \mathrm{K})\alpha \\ g_1\alpha + \mathrm{K} &= g_2\alpha + \mathrm{K} \\ (g_1 + \mathrm{K})\bar{\alpha} &= (g_2 + \mathrm{K})\bar{\alpha}.\end{aligned}$$

Thus $\bar{\alpha}$ is well-defined.

Next we show that $\bar{\alpha}$ is an endomorphism of G_1/K.

Let $\bar{a} = \bar{b}$ in G_1/K , where $\bar{a} = g_1 + \mathrm{K}$, $\bar{b} = g_2 + \mathrm{K}$. Then

$$\begin{aligned}
(\bar{a} + \bar{b})\bar{\alpha} &= ((g_1 + \mathrm{K}) + (g_2 + \mathrm{K}))\bar{\alpha} \\
&= (g_1 + g_2 + \mathrm{K})\bar{\alpha} \\
&= (g_1 + g_2)\alpha + \mathrm{K} \\
&= g_1\alpha + g_2\alpha + \mathrm{K} \\
&= (g_1\alpha + \mathrm{K}) + (g_2\alpha + \mathrm{K}) \\
&= (g_1 + \mathrm{K})\bar{\alpha} + (g_2 + \mathrm{K})\bar{\alpha} \\
&= (\bar{a})\bar{\alpha} + (\bar{b})\bar{\alpha}
\end{aligned}$$

which shows that $\bar{\alpha}$ is an endomorphism of G_1/K.

Let us consider the map

$$\theta : \mathrm{End}(G_1) \longrightarrow \mathrm{End}(G_1/\mathrm{K})$$

defined by

$$(\alpha)\theta = \bar{\alpha}$$

then θ is a homomorphism for if $\alpha, \upsilon \in \mathrm{End}(G_1)$, then

$$\begin{aligned}
(g_1 + \mathrm{K})(\alpha\upsilon)\theta &= (g_1 + \mathrm{K})(\varrho)\theta\,, \text{ where } \varrho = \alpha\upsilon \\
&= (g_1 + \mathrm{K})\bar{\varrho} \\
&= g_1\varrho + \mathrm{K} \\
&= g_1\alpha\upsilon + \mathrm{K} \\
&= (g_1\alpha + \mathrm{K})\bar{\upsilon} \\
&= (g_1 + \mathrm{K})\bar{\alpha}\bar{\upsilon} \\
&= (g_1 + \mathrm{K})(\alpha)\theta(\upsilon)\theta
\end{aligned}$$

and θ is a homomorphism.

Recall that $G_1/\mathrm{K} \cong G_0$; then since $\bar{\alpha}$ is an endomorphism of G_1/K, $\bar{\alpha}$ could be considered as an endomorphism of G_0.

Let Δ be the isomorphism mapping G_1/K to G_0 where

$$\Delta : G_1/\mathrm{K} \longrightarrow G_0$$

is given by

$$(g_1 + \mathrm{K})\Delta = g_1\phi_{1,0}$$

and let ψ be the isomorphism mapping $\mathrm{End}(G_1/\mathrm{K})$ to $\mathrm{End}(G_0)$.

Thus the isomorphism

$$\psi : \mathrm{End}(G_1/\mathrm{K}) \longrightarrow \mathrm{End}(G_0)$$

is defined by

$$\bar{\alpha}\psi = \Delta^{-1}\bar{\alpha}\Delta.$$

From (6.5), we deduce that

$$\bar{\alpha}\psi \;=\; \beta \tag{6.6}$$

and we can write (6.5) as

$$\alpha\phi_{1,0} \;=\; \phi_{1,0}\bar{\alpha}\psi. \tag{6.7}$$

Hence given $\alpha \in \mathrm{End}(G_1)$ such that $\mathrm{K}\alpha \subseteq \mathrm{K}$, we can define a map

$$\alpha^{(1)} : S \longrightarrow S$$

$$(s)\alpha^{(1)} = \begin{cases} s\alpha & \text{if } s \in G_1 , \\ s\bar{\alpha}\psi & \text{if } s \in G_0 \end{cases}$$

From this definition, it is easy to see that $\alpha^{(1)}|_{G1} \in \mathrm{End}(G_1)$ and $\alpha^{(1)}|_{G_0} \in \mathrm{End}(G_0)$. Hence, to show that $\alpha^{(1)}$ is an endomorphism of S, it is sufficient to verify the case in which the two elements of S each lies in a different group. So let us suppose, without loss of generality, that $s_1, s_2 \in S$ such that $s_1 \in G_1$ and $s_2 \in G_0$, (similarly if $s_1 \in G_0$ and $s_2 \in G_1$), then

$$\begin{aligned}
(s_1s_2)\alpha^{(1)} &= (s_0s_2)\alpha^{(1)}, \text{ where } s_0 = (s_1)\phi_{1,0} \in G_0 \\
&= (s_0s_2)\bar{\alpha}\psi \\
&= (s_0\bar{\alpha}\psi)(s_2\bar{\alpha}\psi) \\
&= (s_1\phi_{1,0}\bar{\alpha}\psi)(s_2\bar{\alpha}\psi) \\
&= (s_1\alpha\phi_{1,0})(s_2\bar{\alpha}\psi), \text{ by (6.7)}, \\
&= (s_1\alpha)(s_2\bar{\alpha}\psi) \\
&= (s_1\alpha^{(1)})(s_2\alpha^{(1)}).
\end{aligned}$$

Hence $\alpha^{(1)}$ is an endomorphism of S.

Since $1\alpha^{(1)} = 1$ and $0\alpha^{(1)} = 0$, it follows that $\alpha^{(1)}$ is an endomorphism of type I.

The next step is to consider the endomorphisms of type II on S. In this case, for $f \in \mathrm{End}(S)$, we have $f|_{G_1} \in \mathrm{End}(G_1)$ and $f|_{G_0} \in \mathrm{Hom}(G_0, G_1)$.
Let us suppose that $f|_{G_1} = \alpha$ and $f|_{G_0} = \gamma$, then by theorem 4.2.4, we have

$$(G_1\phi_{1,0})f \;=\; (G_1 f)\phi_{1f,0f}$$

which will imply that

$$\phi_{1,0}\gamma \;=\; \alpha. \tag{6.8}$$

Thus given $\gamma \in \mathrm{Hom}(G_0, G_1)$, we can define a map

$$\gamma^{(2)} : S \longrightarrow S$$

$$(s)\gamma^{(2)} = \begin{cases} s\phi_{1,0}\gamma & \text{if } s \in G_1 , \\ s\gamma & \text{if } s \in G_0. \end{cases}$$

To show that $\gamma^{(2)}$ is an endomorphism of S, it is enough to consider the case when $s_1 \in G_1, s_2 \in G_0$ (similarly when $s_1 \in G_0, s_2 \in G_1$) to get

$$\begin{aligned} (s_1 s_2)\gamma^{(2)} &= (s_0 s_2)\gamma^{(2)}, \text{where } s_0 = (s_1)\phi_{1,0} \in G_0 \\ &= (s_0 s_2)\gamma, \text{since } \gamma = \gamma^{(2)}|_{G_0} \\ &= (s_0\gamma)(s_2\gamma) \\ &= (s_1\phi_{1,0}\gamma)(s_2\gamma) \\ &= (s_1\gamma^{(2)})(s_2\gamma^{(2)}). \end{aligned}$$

Hence $\gamma^{(2)}$ is an endomorphism of S.
Since $1\gamma^{(2)} = 1 = 0\gamma^{(2)}$, it follows that $\gamma^{(2)}$ is an endomorphism of type II.

Finally we consider the endomorphisms of type III on S. For $f \in \mathrm{End}(S)$, we have $f|_{G_1} \in \mathrm{Hom}(G_1, G_0)$ and $f|_{G_0} \in \mathrm{End}(G_0)$.
Suppose that $f|_{G_1} = \delta$ and $f|_{G_0} = \beta$. By theorem 4.2.4, we have

$$(G_1\phi_{1,0})f \;=\; (G_1 f)\phi_{1f,0f}$$

which gives

$$\phi_{1,0}\beta = \delta \tag{6.9}$$

so that $\phi_{1,0}\beta \in \mathrm{Hom}(G_1, G_0)$.

Hence given $\beta \in \mathrm{End}(G_0)$, we can define a map

$$\beta^{(3)} : S \longrightarrow S$$

$$(s)\beta^{(3)} = \begin{cases} s\phi_{1,0}\beta & \text{if } s \in G_1 \ , \\ s\beta & \text{if } s \in G_0 \end{cases}$$

To show that $\beta^{(3)}$ is an endomorphism of S, it is sufficient to consider the case when $s_1 \in G_1, s_2 \in G_0$, (similarly when $s_1 \in G_0$ and $s_2 \in G_1$), to get

$$\begin{aligned}
(s_1 s_2)\beta^{(3)} &= (s_0 s_2)\beta^{(3)}, \text{where } s_0 = (s_1)\phi_{1,0} \in G_0 \\
&= (s_0 s_2)\beta, \text{as } \beta = \beta^{(3)}|_{G_0} \\
&= (s_0\beta)(s_2\beta) \\
&= (s_1\phi_{1,0}\beta)(s_2\beta) \\
&= (s_1\beta^{(3)})(s_2\beta^{(3)}).
\end{aligned}$$

Hence $\beta^{(3)}$ is an endomorphism of S.

It is easily seen that $1\beta^{(3)} = 0 = 0\beta^{(3)}$, which means that $\beta^{(3)}$ is an endomorphism of type III.

We now summarize our observations as follows : given α, an endomorphism of G_1 such that $\mathrm{K}\alpha \subseteq \mathrm{K}$, then α will give rise to an endomorphism of type I on S. Similarly, given $\gamma \in \mathrm{Hom}(G_0, G_1)$ then γ will give rise to an endomorphism of type II on S, while an endomorphism β of G_0 will give rise to an endomorphism of type III.

Next we are going to find out connections between the structures which are generated by these three objects and $E(S)$. In this direction we define

$$\mathrm{End}_K(G_1) := \{\alpha \in \mathrm{End}(G_1) \ ; \ \mathrm{K}\alpha \subseteq \mathrm{K} \ \}$$

and let

$E_K(G_1)$ be the d.g. near-ring generated by $\mathrm{End}_K(G_1)$.

Define a map

$$\Gamma_1 : \mathrm{End}_K(G_1) \longrightarrow \mathrm{End}(S)$$

by

$$(\alpha)\Gamma_1 = \alpha^{(1)}.$$

The map Γ_1 is a homomorphism for, if $\alpha,\ \upsilon \in \mathrm{End}_K(G_1)$, then for $s \in G_1$, we have

$$\begin{aligned} s(\alpha\upsilon)\Gamma_1 &= s(\varrho)\Gamma_1, \text{ where } \varrho = \alpha\upsilon, \\ &= s\varrho^{(1)} \\ &= s\varrho \\ &= s\alpha\upsilon \\ &= s\alpha^{(1)}\upsilon^{(1)} \\ &= s(\alpha\Gamma_1)(\upsilon\Gamma_1), \end{aligned}$$

and for $s \in G_0$, we have

$$\begin{aligned} s(\alpha\upsilon)\Gamma_1 &= s(\varrho)\Gamma_1, \text{ where } \varrho = \alpha\upsilon, \\ &= s\varrho^{(1)} \\ &= s(\bar{\varrho}\psi) \\ &= s(\varrho\theta\psi) \\ &= s((\alpha\upsilon)\theta\psi) \\ &= s(\alpha\theta\psi)(\upsilon\theta\psi) \\ &= s(\bar{\alpha}\psi)(\bar{\upsilon}\psi) \\ &= s(\alpha^{(1)})(\upsilon^{(1)}) \\ &= s(\alpha\Gamma_1)(\upsilon\Gamma_1). \end{aligned}$$

Thus Γ_1 is a homomorphism.

We may extend Γ_1 to the map Γ_1^*, where

$$\Gamma_1^\star : E_K(G_1) \longrightarrow E(S)$$

is given by

$$(\sum \epsilon_i\alpha_i)\Gamma_1^\star = \sum \epsilon_i\alpha_i^{(1)}.$$

where $\epsilon_i = \pm 1$.

We show that $\Gamma_1^\star$ is a homomorphism. For this purpose we will prove that

$$(c)\Gamma_1^\star = 0 \text{ whenever } 0 = c \in E_K(G_1).$$

Let $c \in E_K(G_1)$ such that $c = \sum_{i=1}^n \epsilon_i\alpha_i = 0$, then for $g_1 \in G_1$, we have

$$g_1 \sum_{i=1}^n \epsilon_i\alpha_i \;=\; 0.$$

Thus for $g_1 \in G_1$, we have

$$\begin{aligned}
g_1(c)\Gamma_1^\star &= g_1(\sum_{i=1}^n \epsilon_i\alpha_i)\Gamma_1^\star \\
&= g_1(\sum_{i=1}^n \epsilon_i\alpha_i^{(1)}) \\
&= \sum_{i=1}^n \epsilon_i(g_1\alpha_i^{(1)}) \\
&= \sum_{i=1}^n \epsilon_i(g_1\alpha_i) \\
&= g_1\sum_{i=1}^n \epsilon_i\alpha_i \\
&= 0
\end{aligned}$$

and for $g_0 \in G_0$, we have

$$\begin{aligned}
g_0(c)\Gamma_1^\star &= g_0(\sum_{i=1}^n \epsilon_i\alpha_i)\Gamma_1^\star \\
&= g_0(\sum_{i=1}^n \epsilon_i\alpha_i^{(1)}) \\
&= \sum_{i=1}^n \epsilon_i(g_0\alpha_i^{(1)}) \\
&= \sum_{i=1}^n \epsilon_i(g_0\bar{\alpha}_i\psi) \\
&= \sum_{i=1}^n \epsilon_i(g_1\phi_{1,0})\bar{\alpha}_i\psi \text{ , since } \phi_{1,0} \text{ is an epimorphism ,} \\
&= \sum_{i=1}^n \epsilon_i(g_1\alpha_i\phi_{1,0}), \text{ by (6.7) ,}
\end{aligned}$$

$$= \; g_1(\sum_{i=1}^{n} \epsilon_i\alpha_i)\phi_{1,0}$$
$$= \; 0.$$

Hence $(c)\Gamma_1^\star = 0$ and $\Gamma_1^\star$ is a homomorphism.

Furthermore, we show that $\Gamma_1^\star$ is a monomorphism.

Suppose that $c_1 \neq c_2$ in $E_K(G_1)$, where $c_1 = \sum_{i=1}^{n} \epsilon_i\alpha_i$ and $c_2 = \sum_{j=1}^{r} \eta_j\upsilon_j$, then there exists $g_1 \in G_1$ such that

$$\begin{aligned}
g_1\sum_{i=1}^{n}\epsilon_i\alpha_i &\neq g_1\sum_{j=1}^{r}\eta_j\upsilon_j \\
\sum_{i=1}^{n}\epsilon_i g_1\alpha_i &\neq \sum_{j=1}^{r}\eta_j g_1\upsilon_j \\
\sum_{i=1}^{n}\epsilon_i g_1\alpha_i^{(1)} &\neq \sum_{j=1}^{r}\eta_j g_1\upsilon_j^{(1)} \\
g_1\sum_{i=1}^{n}\epsilon_i\alpha_i^{(1)} &\neq g_1\sum_{j=1}^{r}\eta_j\upsilon_j^{(1)} \\
g_1(\sum_{i=1}^{n}\epsilon_i\alpha_i)\Gamma_1^\star &\neq g_1(\sum_{j=1}^{r}\eta_j\upsilon_j)\Gamma_1^\star \\
(c_1)\Gamma_1^\star &\neq (c_2)\Gamma_1^\star.
\end{aligned}$$

Hence $\Gamma_1^\star$ is a monomorphism.

This shows that the endomorphisms of type I on S will generate in $E(S)$ a subnear-ring isomorphic to the near-ring $E_K(G_1)$.

Next we consider the set $\mathrm{Hom}(G_0, G_1)$ and define a map Γ_2, where

$$\Gamma_2 : \mathrm{Hom}(G_0, G_1) \longrightarrow \mathrm{End}(S)$$

is given by

$$(\gamma)\Gamma_2 = \gamma^{(2)}.$$

We show that Γ_2 is a 1-1 map.

Let $\gamma, \vartheta \in \mathrm{Hom}(G_0, G_1)$ such that $\gamma \neq \vartheta$, then there exists $g_0 \in G_0$ such that

$$\begin{aligned}
g_0\gamma &\neq g_0\vartheta, \\
g_0\gamma^{(2)} &\neq g_0\vartheta^{(2)}, \\
g_0(\gamma)\Gamma_2 &\neq g_0(\vartheta)\Gamma_2
\end{aligned}$$

which shows that $(\gamma)\Gamma_2 \neq (\vartheta)\Gamma_2$ and Γ_2 is 1-1.

Now we define the group $(\text{gp}\!<\mathrm{Hom}(G_0,G_1)>,+)$ and extend Γ_2 to the map

$$\Gamma_2^\star : (\text{gp}\!<\mathrm{Hom}(G_0,G_1)>,+) \longrightarrow E(S)$$

which is given by

$$(\sum \epsilon_i\gamma_i)\Gamma_2^\star = \sum \epsilon_i\gamma_i^{(2)}.$$

where $\epsilon_i = \pm 1$.

We show that the map $\Gamma_2^\star$ is a homomorphism.

First we can see that

$$(\gamma+\vartheta)\Gamma_2^\star = (\gamma)\Gamma_2^\star + (\vartheta)\Gamma_2^\star$$

since for $g_1 \in G_1$, we have

$$\begin{aligned}
g_1(\gamma+\vartheta)\Gamma_2^\star &= g_1(\varepsilon)\Gamma_2^\star, \text{where } \varepsilon = \gamma+\vartheta\\
&= g_1\varepsilon^{(2)}\\
&= g_1\phi_{1,0}\varepsilon\\
&= g_1\phi_{1,0}(\gamma+\vartheta)\\
&= g_1\phi_{1,0}\gamma + g_1\phi_{1,0}\vartheta\\
&= g_1\gamma^{(2)} + g_1\vartheta^{(2)}\\
&= g_1(\gamma^{(2)}+\vartheta^{(2)})\\
&= g_1((\gamma)\Gamma_2^\star + (\vartheta)\Gamma_2^\star),
\end{aligned}$$

and for $g_0 \in G_0$, we have

$$\begin{aligned}
g_0(\gamma+\vartheta)\Gamma_2^\star &= g_0(\varepsilon)\Gamma_2^\star\,,\ \varepsilon = \gamma+\vartheta,\\
&= g_0\varepsilon^{(2)}\\
&= g_0\varepsilon\\
&= g_0(\gamma+\vartheta)\\
&= g_0\gamma + g_0\vartheta\\
&= g_0\gamma^{(2)} + g_0\vartheta^{(2)}\\
&= g_0(\gamma^{(2)}+\vartheta^{(2)})\\
&= g_0((\gamma)\Gamma_2^\star + (\vartheta)\Gamma_2^\star).
\end{aligned}$$

Thus, to say that $\Gamma_2^\star$ is a homomorphism, we only need to show that

$$(c)\Gamma_2^\star = 0 \text{ whenever } 0 = c \in (\text{gp}< \text{Hom}(G_0, G_1) >, +).$$

Suppose that $0 = c = \sum_{i=1}^n \epsilon_i\gamma_i \in (\text{gp}< \text{Hom}(G_0, G_1) >, +)$. Then for all $g_0 \in G_0$,

$$g_0 \sum_{i=1}^n \epsilon_i\gamma_i = 0.$$

So for $g_1 \in G_1$, we have

$$\begin{aligned} g_1(c)\Gamma_2^\star &= g_1(\sum_{i=1}^n \epsilon_i\gamma_i)\Gamma_2^\star \\ &= g_1\sum_{i=1}^n \epsilon_i\gamma_i^{(2)} \\ &= \sum_{i=1}^n \epsilon_i g_1\gamma_i^{(2)} \\ &= \sum_{i=1}^n \epsilon_i g_1\phi_{1,0}\gamma_i \\ &= \sum_{i=1}^n \epsilon_i g_0\gamma_i, \text{ where } g_0 = g_1\phi_{1,0}, \\ &= g_0\sum_{i=1}^n \epsilon_i\gamma_i \\ &= 0 \end{aligned}$$

and if $g_0 \in G_0$, then

$$\begin{aligned} g_0(c)\Gamma_2^\star &= g_0(\sum_{i=1}^n \epsilon_i\gamma_i)\Gamma_2^\star \\ &= g_0\sum_{i=1}^n \epsilon_i\gamma_i^{(2)} \\ &= g_0\sum_{i=1}^n \epsilon_i\gamma_i \\ &= 0. \end{aligned}$$

Hence $(c)\Gamma_2^\star = 0$ and $\Gamma_2^\star$ is a homomorphism.

Moreover, $\Gamma_2^\star$ is a monomorphism, since if we suppose that $c_1 \neq c_2$ in (gp< $\text{Hom}(G_0, G_1)$ >, +), where $c_1 = \sum_{i=1}^n \epsilon_i\gamma_i, c_2 = \sum_{j=1}^r \eta_j\zeta_j$, then there exists $g_0 \in G_0$ such that

$$g_0\sum_{i=1}^n \epsilon_i\gamma_i \neq g_0\sum_{j=1}^r \eta_j\zeta_j$$

$$
\begin{aligned}
\sum_{i=1}^{n} \epsilon_i g_0 \gamma_i &\neq \sum_{j=1}^{r} \eta_j g_0 \zeta_j \\
\sum_{i=1}^{n} \epsilon_i g_0 \gamma_i^{(2)} &\neq \sum_{j=1}^{r} \eta_j g_0 \zeta_j^{(2)} \\
g_0 \sum_{i=1}^{n} \epsilon_i \gamma_i^{(2)} &\neq g_0 \sum_{j=1}^{r} \eta_j \zeta_j^{(2)} \\
g_0 (\sum_{i=1}^{n} \epsilon_i \gamma_i) \Gamma_2^\star &\neq g_0 (\sum_{j=1}^{r} \eta_j \zeta_j) \Gamma_2^\star \\
(c_1)\Gamma_2^\star &\neq (c_2)\Gamma_2^\star.
\end{aligned}
$$

Hence $\Gamma_2^\star$ is a monomorphism.

We deduce that there is an isomorphic copy of the group $(\mathrm{gp}< \mathrm{Hom}(G_0, G_1) >, +)$ inside $E(S)$.

Finally we consider the endomorphisms of G_0 and define a map Γ_3, where

$$\Gamma_3 : \mathrm{End}(G_0) \longrightarrow \mathrm{End}(S)$$

is given by

$$(\beta)\Gamma_3 = \beta^{(3)}.$$

The map Γ_3 is a homomorphism, for if $\beta, \xi \in \mathrm{End}(G_0)$, then for $g_1 \in G_1$, we have

$$
\begin{aligned}
g_1(\beta\xi)\Gamma_3 &= g_1(\nu)\Gamma_3, \text{where } \nu = \beta\xi, \\
&= g_1 \nu^{(3)} \\
&= g_1 \phi_{1,0} \nu \\
&= g_1 \phi_{1,0} \beta \xi \\
&= g_1 (\phi_{1,0} \beta) \xi \\
&= g_1 \beta^{(3)} \xi \\
&= g_1 \beta^{(3)} \xi^{(3)} \\
&= g_1 (\beta\Gamma_3)(\xi\Gamma_3),
\end{aligned}
$$

and for $g_0 \in G_0$, we have

$$g_0(\beta\xi)\Gamma_3 = g_0(\nu)\Gamma_3, \text{where } \nu = \beta\xi,$$

$$
\begin{aligned}
&= g_0\nu^{(3)} \\
&= g_0\nu \\
&= g_0\beta\xi \\
&= g_0\beta^{(3)}\xi^{(3)} \\
&= g_0(\beta\Gamma_3)(\xi\Gamma_3).
\end{aligned}
$$

Thus Γ_3 is a homomorphism.

Now we can extend Γ_3 to the map $\Gamma_3^\star$ such that

$$\Gamma_3^\star : E(G_0) \longrightarrow E(S)$$

is given by

$$(\sum \epsilon_i\beta_i)\Gamma_3^\star = \sum \epsilon_i\beta_i^{(3)}.$$

where $\epsilon_i = \pm 1$.

We show that $\Gamma_3^\star$ is a homomorphism by showing that

$$(c)\Gamma_3^\star = 0 \text{ whenever } 0 = c \in E(G_0).$$

Let $c \in E(G_0)$ such that $c = \sum_{i=1}^n \epsilon_i\beta_i = 0$, then $g_0 \sum_{i=1}^n \epsilon_i\beta_i = 0$ for all $g_0 \in G_0$. Thus for $g_1 \in G_1$, we have

$$
\begin{aligned}
g_1(c)\Gamma_3^\star &= g_1(\sum_{i=1}^n \epsilon_i\beta_i)\Gamma_3^\star \\
&= g_1(\sum_{i=1}^n \epsilon_i\beta_i^{(3)}) \\
&= \sum_{i=1}^n \epsilon_i(g_1\beta_i^{(3)}) \\
&= \sum_{i=1}^n \epsilon_i(g_1\phi_{1,0}\beta_i) \\
&= \sum_{i=1}^n \epsilon_i(g_0\beta_i), \text{where } g_0 = g_1\phi_{1,0}, \\
&= g_0\sum_{i=1}^n \epsilon_i\beta_i \\
&= 0
\end{aligned}
$$

and for $g_0 \in G_0$, we have

$$\begin{aligned}
g_0(c)\Gamma_3^\star &= g_0(\sum_{i=1}^{n}\epsilon_i\beta_i)\Gamma_3^\star \\
&= g_0(\sum_{i=1}^{n}\epsilon_i\beta_i^{(3)}) \\
&= \sum_{i=1}^{n}\epsilon_i(g_0\beta_i^{(3)}) \\
&= \sum_{i=1}^{n}\epsilon_i(g_0\beta_i) \\
&= g_0\sum_{i=1}^{n}\epsilon_i\beta_i \\
&= 0.
\end{aligned}$$

Hence $(c)\Gamma_3^\star = 0$ and $\Gamma_3^\star$ is a homomorphism.

Furthermore, $\Gamma_3^\star$ is a monomorphism. To see that let us assume that $c_1 \neq c_2$ in $E(G_0)$, where $c_1 = \sum_{i=1}^{n}\epsilon_i\beta_i, c_2 = \sum_{j=1}^{r}\eta_j\nu_j$. Then there exists $g_0 \in G_0$ such that

$$\begin{aligned}
g_0\sum_{i=1}^{n}\epsilon_i\beta_i &\neq g_0\sum_{j=1}^{r}\eta_j\nu_j \\
\sum_{i=1}^{n}\epsilon_i g_0\beta_i &\neq \sum_{j=1}^{r}\eta_j g_0\nu_j \\
\sum_{i=1}^{n}\epsilon_i g_0\beta_i^{(3)} &\neq \sum_{j=1}^{r}\eta_j g_0\nu_j^{(3)} \\
g_0\sum_{i=1}^{n}\epsilon_i\beta_i^{(3)} &\neq g_0\sum_{j=1}^{r}\eta_j\nu_j^{(3)} \\
g_0(\sum_{i=1}^{n}\epsilon_i\beta_i)\Gamma_3^\star &\neq g_0(\sum_{j=1}^{r}\eta_j\nu_j)\Gamma_3^\star \\
(c_1)\Gamma_3^\star &\neq (c_2)\Gamma_3^\star.
\end{aligned}$$

Hence $\Gamma_3^\star$ is a monomorphism.

This shows that the endomorphisms of type III on S will generate in $E(S)$ a subnear-ring isomorphic to the near-ring $E(G_0)$.

From above we can write

$$E(S) = E_K(G_1)^I \cup E(G_0, G_1)^{II} \cup E(G_0)^{III} \tag{6.10}$$

where $E_K(G_1)^I$ is the copy of the near-ring $E_K(G_1)$ in $E(S)$ which is generated by the endomorphisms of type I on S, and $E(G_0, G_1)^{II}$ is an isomorphic copy of the group (gp< $\mathrm{Hom}(G_0, G_1)$ >, +) in $E(S)$, while $E(G_0)^{III}$ is the copy of the near-ring $E(G_0)$ which is generated by the endomorphisms of type III.

6.2 Addition in $E(S)$

In this section we are going to describe the sum of elements in $E(S)$. For this purpose we need first to study the sum of endomorphisms of different types, then we proceed and consider relation (6.10). To do so, we have to start with the following :

Proposition 6.2.1 With the same notation as in section 6.1, for $\phi_{1,0}\gamma \in \mathrm{End}(G_1)$, where $\gamma \in \mathrm{Hom}(G_0, G_1)$, the following equation is satisfied

$$(\overline{\phi_{1,0}\gamma})\psi \;=\; \gamma\phi_{1,0} \tag{6.11}$$

Proof Consider $g_0 \in G_0$ then

$$\begin{aligned}
g_0(\overline{\phi_{1,0}\gamma})\psi &= g_0(\bar{\alpha})\psi, \text{ where } \alpha = \phi_{1,0}\gamma \\
&= g_0\Delta^{-1}\bar{\alpha}\Delta \\
&= (g_1\phi_{1,0}\Delta^{-1})\bar{\alpha}\Delta \text{ , where } g_1\phi_{1,0} = g_0 \text{ ,} \\
&= (g_1 + \mathrm{K})\bar{\alpha}\Delta \\
&= g_1\alpha\phi_{1,0} \\
&= g_1\phi_{1,0}\gamma\phi_{1,0} \\
&= g_0\gamma\phi_{1,0}
\end{aligned}$$

Hence (6.11) is satisfied.

Now we first consider the sum $\alpha^{(1)} + \gamma^{(2)}$ of two endomorphisms of type I and II respectively. For $g_1 \in G_1$, we have

$$\begin{aligned}
g_1(\alpha^{(1)} + \gamma^{(2)}) &= g_1\alpha^{(1)} + g_1\gamma^{(2)} \\
&= g_1\alpha^{(1)} + g_1\phi_{1,0}\gamma \\
&= g_1\alpha^{(1)} + g_1(\phi_{1,0}\gamma)^{(1)} \\
&= g_1(\alpha^{(1)} + (\phi_{1,0}\gamma)^{(1)}),
\end{aligned} \tag{6.12}$$

and for $g_0 \in G_0$, we have

$$\begin{aligned}
g_0(\alpha^{(1)} + \gamma^{(2)}) &= g_0\alpha^{(1)} + g_0\gamma^{(2)} \\
&= g_0(\bar{\alpha}\psi) + g_0\gamma \\
&= g_0(\bar{\alpha}\psi) + g_0\gamma\phi_{1,0} \\
&= g_0(\bar{\alpha}\psi) + g_0(\overline{\phi_{1,0}\gamma})\psi, \text{ by proposition (6.2.1)} \\
&= g_0\alpha^{(1)} + g_0(\phi_{1,0}\gamma)^{(1)} \\
&= g_0(\alpha^{(1)} + (\phi_{1,0}\gamma)^{(1)}). \qquad (6.13)
\end{aligned}$$

Equations (6.12) and (6.13) imply that

$$\alpha^{(1)} + \gamma^{(2)} = \alpha^{(1)} + (\phi_{1,0}\gamma)^{(1)}. \qquad (6.14)$$

Similarly we can show that

$$\gamma^{(2)} + \alpha^{(1)} = (\phi_{1,0}\gamma)^{(1)} + \alpha^{(1)}. \qquad (6.15)$$

Hence the sum of two endomorphisms of type I and II could be considered as a sum of endomorphisms of type I.

Next we consider the sum $\alpha^{(1)} + \beta^{(3)}$ of two endomorphisms of type I and III respectively. For $g_1 \in G_1$, we have

$$\begin{aligned}
g_1(\alpha^{(1)} + \beta^{(3)}) &= g_1\alpha^{(1)} + g_1\beta^{(3)} \\
&= g_1\alpha + g_1\beta^{(3)} \\
&= g_1\alpha\phi_{1,0} + g_1\beta^{(3)} \\
&= g_1\phi_{1,0}\bar{\alpha}\psi + g_1\beta^{(3)}, \text{by (6.7)} \\
&= g_1(\bar{\alpha}\psi)^{(3)} + g_1\beta^{(3)} \\
&= g_1((\bar{\alpha}\psi)^{(3)} + \beta^{(3)}), \qquad (6.16)
\end{aligned}$$

and for $g_0 \in G_0$, we have

$$\begin{aligned}
g_0(\alpha^{(1)} + \beta^{(3)}) &= g_0\alpha^{(1)} + g_0\beta^{(3)} \\
&= g_0\bar{\alpha}\psi + g_0\beta^{(3)}
\end{aligned}$$

$$
\begin{aligned}
&= g_0(\bar{\alpha}\psi)^{(3)} + g_0\beta^{(3)} \\
&= g_0((\bar{\alpha}\psi)^{(3)} + \beta^{(3)}). \qquad (6.17)
\end{aligned}
$$

Equations (6.16) and (6.17) show that

$$
\alpha^{(1)} + \beta^{(3)} = (\bar{\alpha}\psi)^{(3)} + \beta^{(3)}. \qquad (6.18)
$$

Similarly we can get

$$
\beta^{(3)} + \alpha^{(1)} = \beta^{(3)} + (\bar{\alpha}\psi)^{(3)} \qquad (6.19)
$$

which shows that the sum of two endomorphisms of type I and type III could be regarded as a sum of endomorphisms of type III.

Finally we consider the sum $\gamma^{(2)} + \beta^{(3)}$ of endomorphisms of type II and III respectively. For $g_1 \in G_1$, we have

$$
\begin{aligned}
g_1(\gamma^{(2)} + \beta^{(3)}) &= g_1\gamma^{(2)} + g_1\beta^{(3)} \\
&= g_1\phi_{1,0}\gamma + g_1\phi_{1,0}\beta \\
&= g_1\phi_{1,0}\gamma\phi_{1,0} + g_1\phi_{1,0}\beta \\
&= g_1(\gamma\phi_{1,0})^{(3)} + g_1\beta^{(3)} \\
&= g_1((\gamma\phi_{1,0})^{(3)} + \beta^{(3)}), \qquad (6.20)
\end{aligned}
$$

and for $g_0 \in G_0$, we have

$$
\begin{aligned}
g_0(\gamma^{(2)} + \beta^{(3)}) &= g_0\gamma^{(2)} + g_0\beta^{(3)} \\
&= g_0\gamma + g_0\beta \\
&= g_0\gamma\phi_{1,0} + g_0\beta \\
&= g_0(\gamma\phi_{1,0})^{(3)} + g_0\beta^{(3)} \\
&= g_0((\gamma\phi_{1,0})^{(3)} + \beta^{(3)}). \qquad (6.21)
\end{aligned}
$$

Equations (6.20) and (6.21) imply that

$$
\gamma^{(2)} + \beta^{(3)} = (\gamma\phi_{1,0})^{(3)} + \beta^{(3)} \qquad (6.22)
$$

Similarly we can get

$$\beta^{(3)} + \gamma^{(2)} \;=\; \beta^{(3)} + (\gamma\phi_{1,0})^{(3)} \tag{6.23}$$

so that the sum of two endomorphisms of type II and type III could be considered as a sum of endomorphisms of type III.

In order to get a precise description of the sum in $E(S)$, we use induction and follow the same pattern as in chapter 1 using equations (6.14), (6.18) and (6.22) to get

$$\sum_{i=1}^{n}\epsilon_i\alpha_i^{(1)} + \sum_{j=1}^{k}\eta_j\gamma_j^{(2)} \;=\; \sum_{i=1}^{n}\epsilon_i\alpha_i^{(1)} + \sum_{j=1}^{k}\eta_j(\phi_{1,0}\gamma_j)^{(1)} \tag{6.24}$$

$$\sum_{i=1}^{n}\epsilon_i\alpha_i^{(1)} + \sum_{r=1}^{m}\delta_r\beta_r^{(3)} \;=\; \sum_{i=1}^{n}\epsilon_i(\bar{\alpha}_i\psi)^{(3)} + \sum_{r=1}^{m}\delta_r\beta_r^{(3)} \tag{6.25}$$

$$\sum_{j=1}^{k}\eta_j\gamma_j^{(2)} + \sum_{r=1}^{m}\delta_r\beta_r^{(3)} \;=\; \sum_{j=1}^{k}\eta_j(\gamma_j\phi_{1,0})^{(3)} + \sum_{r=1}^{m}\delta_r\beta_r^{(3)} \tag{6.26}$$

where,

$(\,)^{(1)} \in$ type I , $(\,)^{(2)} \in$ type II , $(\,)^{(3)} \in$ type III and $\epsilon_i = \pm 1$, $\eta_j = \pm 1$, $\delta_r = \pm 1$, $i = 1, \ldots, n$, $r = 1, \ldots, m$, $j = 1, \ldots, k$.

We note that the same conclusion can be obtained if we reverse the order of addition in the above equations. Hence, we observe that the sum of an element in $E_K(G_1)^I$ with an element in $E(G_0, G_1)^{II}$ lies in $E_K(G_1)^I$, and the sum of an element in $E_K(G_1)^I$ with an element in $E(G_0)^{III}$ lies in $E(G_0)^{III}$, and the sum of an element in $E(G_0, G_1)^{II}$ with an element in $E(G_0)^{III}$ lies in $E(G_0)^{III}$. We summarize this sum as follows :

$E_K(G_1)^I + E(G_0, G_1)^{II} \longrightarrow E_K(G_1)^I$ and $E(G_0, G_1)^{II} + E_K(G_1)^I \longrightarrow E_K(G_1)^I$

$E_K(G_1)^I + E(G_0)^{III} \longrightarrow E(G_0)^{III}$ and $E(G_0)^{III} + E_K(G_1)^I \longrightarrow E(G_0)^{III}$

$E(G_0, G_1)^{II} + E(G_0)^{III} \longrightarrow E(G_0)^{III}$ and $E(G_0)^{III} + E(G_0, G_1)^{II} \longrightarrow E(G_0)^{III}$.

Now we turn to find the product in $E(S)$.

6.3 Product in $E(S)$

The endomorphisms of each type are closed under product. This is clear for endomorphisms of type I and type III. Endomorphisms of type II are also closed under product since if $\gamma^{(2)}$ and $\nu^{(2)}$ are two endomorphisms of type II, then for $g_1 \in G_1$, we have

$$\begin{aligned}(g_1)\gamma^{(2)}\nu^{(2)} &= (g_1\phi_{1,0}\gamma)\nu^{(2)} \\ &= g_1\phi_{1,0}\gamma\phi_{1,0}\nu \\ &= g_1(\gamma\phi_{1,0}\nu)^{(2)},\end{aligned}$$

and for $g_0 \in G_0$, we have

$$\begin{aligned}(g_0)\gamma^{(2)}\nu^{(2)} &= (g_0\gamma)\nu^{(2)} \\ &= g_0\gamma\phi_{1,0}\nu \\ &= g_0(\gamma\phi_{1,0}\nu)^{(2)}.\end{aligned}$$

So

$$\gamma^{(2)}\nu^{(2)} = (\gamma\phi_{1,0}\nu)^{(2)}$$

and the endomorphisms of type II are closed under product. Now we look at the product of endomorphisms of different types. Let $\alpha^{(1)}, \gamma^{(2)}$ be endomorphisms of type I and type II respectively. Then for $g_1 \in G_1$, we have

$$\begin{aligned}g_1(\alpha^{(1)}\gamma^{(2)}) &= (g_1\alpha)\gamma^{(2)} \\ &= g_1\alpha\phi_{1,0}\gamma \\ &= g_1\phi_{1,0}\bar{\alpha}\psi\gamma, \text{by (6.7)} \\ &= g_1((\bar{\alpha}\psi)\gamma)^{(2)},\end{aligned} \tag{6.27}$$

and for $g_0 \in G_0$, we have

$$\begin{aligned}g_0(\alpha^{(1)}\gamma^{(2)}) &= (g_0\bar{\alpha}\psi)\gamma^{(2)} \\ &= g_0\bar{\alpha}\psi\gamma \\ &= g_0((\bar{\alpha}\psi)\gamma)^{(2)}.\end{aligned} \tag{6.28}$$

Equations (6.27) and (6.28) imply that

$$\alpha^{(1)}\gamma^{(2)} = ((\bar{\alpha}\psi)\gamma)^{(2)} \tag{6.29}$$

which shows that the product $\alpha^{(1)}\gamma^{(2)}$ could be considered as a product of endomorphisms of type II.

Now we consider the above product when reversing the order of the maps to get, for $g_1 \in G_1$

$$\begin{aligned} g_1(\gamma^{(2)}\alpha^{(1)}) &= (g_1\phi_{1,0}\gamma)\alpha^{(1)} \\ &= g_1\phi_{1,0}\gamma\alpha \\ &= g_1(\gamma\alpha)^{(2)}, \end{aligned} \tag{6.30}$$

and for $g_0 \in G_0$, we have

$$\begin{aligned} g_0(\gamma^{(2)}\alpha^{(1)}) &= (g_0\gamma)\alpha^{(1)} \\ &= g_0\gamma\alpha \\ &= g_0(\gamma\alpha)^{(2)}. \end{aligned} \tag{6.31}$$

Equations (6.30) and (6.31) imply that

$$\gamma^{(2)}\alpha^{(1)} = (\gamma\alpha)^{(2)} \tag{6.32}$$

so that the product $\gamma^{(2)}\alpha^{(1)}$ could be considered as a product of endomorphisms of type II.

Next we consider the product $\alpha^{(1)}\beta^{(3)}$. For $g_1 \in G_1$, we have

$$\begin{aligned} g_1(\alpha^{(1)}\beta^{(3)}) &= (g_1\alpha)\beta^{(3)} \\ &= g_1\alpha\phi_{1,0}\beta \\ &= g_1\phi_{1,0}\bar{\alpha}\psi\beta, \text{by (6.7)} \\ &= g_1((\bar{\alpha}\psi)\beta)^{(3)}, \end{aligned} \tag{6.33}$$

and for $g_0 \in G_0$, we have

$$\begin{aligned} g_0(\alpha^{(1)}\beta^{(3)}) &= (g_0\bar{\alpha}\psi)\beta^{(3)} \\ &= g_0(\bar{\alpha}\psi)\beta \\ &= g_0((\bar{\alpha}\psi)\beta)^{(3)}. \end{aligned} \tag{6.34}$$

Equations (6.33) and (6.34) imply that

$$\alpha^{(1)}\beta^{(3)} = ((\bar{\alpha}\psi)\beta)^{(3)} \tag{6.35}$$

that is the product $\alpha^{(1)}\beta^{(3)}$ could be considered as a product of endomorphisms of type III.

If we reverse the order of the maps in the above product, then for $g_1 \in G_1$, we get

$$\begin{aligned} g_1(\beta^{(3)}\alpha^{(1)}) &= (g_1\phi_{1,0}\beta)\alpha^{(1)} \\ &= (g_1\phi_{1,0}\beta)\bar{\alpha}\psi \\ &= g_1(\beta(\bar{\alpha}\psi))^{(3)}, \end{aligned} \tag{6.36}$$

and for $g_0 \in G_0$, we have

$$\begin{aligned} g_0(\beta^{(3)}\alpha^{(1)}) &= (g_0\beta)\alpha^{(1)} \\ &= g_0(\beta(\bar{\alpha}\psi)) \\ &= g_0(\beta(\bar{\alpha}\psi))^{(3)}. \end{aligned} \tag{6.37}$$

Equations (6.36) and (6.37) give

$$\beta^{(3)}\alpha^{(1)} = (\beta(\bar{\alpha}\psi))^{(3)} \tag{6.38}$$

which shows that the product $\beta^{(3)}\alpha^{(1)}$ could be assumed as a product of endomorphisms of type III.

Finally we consider the product $\gamma^{(2)}\beta^{(3)}$. For $g_1 \in G_1$, we have

$$\begin{aligned} g_1(\gamma^{(2)}\beta^{(3)}) &= (g_1\phi_{1,0}\gamma)\beta^{(3)} \\ &= g_1\phi_{1,0}\gamma\phi_{1,0}\beta \\ &= g_1(\gamma\phi_{1,0}\beta)^{(3)}, \end{aligned} \tag{6.39}$$

and for $g_0 \in G_0$, we have

$$\begin{aligned} g_0(\gamma^{(2)}\beta^{(3)}) &= (g_0\gamma)\beta^{(3)} \\ &= g_0\gamma\phi_{1,0}\beta \\ &= g_0(\gamma\phi_{1,0}\beta)^{(3)}. \end{aligned} \tag{6.40}$$

Equations (6.39) and (6.40) imply that

$$\gamma^{(2)}\beta^{(3)} = (\gamma\phi_{1,0}\beta)^{(3)} \tag{6.41}$$

which shows that the product $\gamma^{(2)}\beta^{(3)}$ could be regarded as a product of endomorphisms of type III.

But if we reverse the order of the maps in the above product, we get

$$\begin{aligned} g_1(\beta^{(3)}\gamma^{(2)}) &= (g_1\phi_{1,0}\beta)\gamma^{(2)} \\ &= g_1\phi_{1,0}\beta\gamma \\ &= g_1(\beta\gamma)^{(2)}, \end{aligned} \tag{6.42}$$

and for $g_0 \in G_0$, we have

$$\begin{aligned} g_0(\beta^{(3)}\gamma^{(2)}) &= (g_0\beta)\gamma^{(2)} \\ &= g_0\beta\gamma \\ &= g_0(\beta\gamma)^{(2)}. \end{aligned} \tag{6.43}$$

Equations (6.42) and (6.43) imply that

$$\beta^{(3)}\gamma^{(2)} = (\beta\gamma)^{(2)} \tag{6.44}$$

which gives a different result from equation (6.41). So the product fails to construct a semilattice of groups. We summarize the product as follows

$E_K(G_1)^I.E(G_0,G_1)^{II} \longrightarrow E(G_0,G_1)^{II}$ and $E(G_0,G_1)^{II}.E_K(G_1)^I \longrightarrow E(G_0,G_1)^{II}$

$E_K(G_1)^I.E(G_0)^{III} \longrightarrow E(G_0)^{III}$ and $E(G_0)^{III}.E_K(G_1)^I \longrightarrow E(G_0)^{III}$

$E(G_0,G_1)^{II}.E(G_0)^{III} \longrightarrow E(G_0)^{III}$ but $E(G_0)^{III}.E(G_0,G_1)^{II} \longrightarrow E(G_0,G_1)^{II}$.

Our conclusion will appear in the following.

6.4 Conclusion II

Let

$$A = E_K(G_1)^I,\ B = E(G_0,G_1)^{II},\ C = E(G_0)^{III}$$

then relation (6.10) can be written as

$$E(S) \;=\; A \cup B \cup C. \tag{6.45}$$

Let a, b and c be the zero maps in A, B and C respectively, so we have the semilattice

$$\mathcal{L} = \{a, b, c\}, \text{with } c \leq a \leq b.$$

Define the linking homomorphisms $\phi_{b,a}, \phi_{b,c}$ and $\phi_{a,c}$ as follows

$$\begin{gathered}\phi_{b,a} : B \longrightarrow A \\ \textstyle\sum_j \eta_j \gamma_j^{(2)} \longrightarrow \sum_j \eta_j (\phi_{1,0}\gamma_j)^{(1)},\end{gathered}$$

$$\begin{gathered}\phi_{b,c} : B \longrightarrow C \\ \textstyle\sum_j \eta_j \gamma_j^{(2)} \longrightarrow \sum_j \eta_j (\gamma_j \phi_{1,0})^{(3)},\end{gathered}$$

$$\begin{gathered}\phi_{a,c} : A \longrightarrow C \\ \textstyle\sum_i \epsilon_i \alpha_i^{(1)} \longrightarrow \sum_i \epsilon_i (\bar{\alpha}_i \psi)^{(3)}.\end{gathered}$$

Hence we have a strong semilattice of groups given by

$$E(S) = (\mathcal{L}\ , \{A, B, C\}, \{\phi_{b,a}, \phi_{b,c}, \phi_{a,c}\})$$

that is,

$(E(S), +)$ is a Clifford semigroup.

Chapter 7

Seminear-ring of endomorphisms III

7.1 Starting case III

Recall that in the previous chapter we considered the groups G_1 and G_0 which were linked by an epimorphism. In this chapter we weaken this relation so that our groups are linked by a homomorphism which may be neither one-to-one nor onto. Again we start with a semilattice

$$Y = \{0, 1\}, \text{ with } 0 \leq 1.$$

Suppose that G_1 and G_0 are two groups which are linked by a homomorphism (not necessarily onto or 1-1)

$$\phi_{1,0} : G_1 \longrightarrow G_0.$$

Consider $S = G_1 \bigcup G_0$. We note that the endomorphisms of type II and type III here are similar to case II in chapter 6, therefore we will be discussing only type I in detail. So let us suppose that f is an endomorphism of type I on S, then by theorem 4.2.3, we have

$f|_{G_1} = \alpha$, say, where $\alpha \in \text{End}(G_1)$ and $f|_{G_0} = \beta$, say, where $\beta \in \text{End}(G_0)$.

Let K $=$ Ker $\phi_{1,0}$ then, as in the preceeding chapter, we have

$$(g_1 + k)\alpha = g_1\alpha + k', \ k, k' \in \text{K},$$

and α induces an endomorphism $\bar{\alpha}$ on G_1/K where

$$\bar{\alpha} : G_1/\text{K} \longrightarrow G_1/\text{K}$$

is given by

$$(g_1 + \mathrm{K})\bar{\alpha} \;=\; g_1\alpha + \mathrm{K}$$

We notice that equation (6.5) is still valid.

Let

$$\overline{G_0} := \mathrm{Im}\phi_{1,0}$$

then

$$\overline{G_0} \cong G_1/\mathrm{K}$$

By (6.5), in this case, β must satisfy the relation

$$\overline{G_0}\beta \;\subseteq\; \overline{G_0}. \tag{7.1}$$

Since $\bar{\alpha} \in \mathrm{End}(G_1/\mathrm{K})$, $\bar{\alpha}$ could be considered as an endomorphism of $\overline{G_0}$; there is also the fact that any endomorphism of G_0 arising from $\mathrm{End}(S)$ must satisfy (7.1).

Consider the isomorphism

$$\Delta : G_1/\mathrm{K} \longrightarrow \overline{G_0}$$

given by

$$(g_1 + \mathrm{K})\Delta = g_1\phi_{1,0}$$

then we can define the isomorphism

$$\psi : \mathrm{End}(G_1/\mathrm{K}) \longrightarrow \mathrm{End}(\overline{G_0})$$

by

$$\bar{\alpha}\psi = \Delta^{-1}\bar{\alpha}\Delta.$$

Let $g_0 \in \overline{G_0}$, then

$$\begin{aligned} g_0\bar{\alpha}\psi &= g_0\Delta^{-1}\bar{\alpha}\Delta \\ &= g_1\phi_{1,0}\Delta^{-1}\bar{\alpha}\Delta, \text{ for some } g_1 \text{ such that } g_1\phi_{1,0} = g_0\,, \\ &= (g_1 + \mathrm{K})\bar{\alpha}\Delta \\ &= (g_1\alpha + \mathrm{K})\Delta \\ &= g_1\alpha\phi_{1,0}. \end{aligned} \tag{7.2}$$

On the other hand for $g_0 \in \overline{G_0}$, we have

$$\begin{aligned} g_0\beta &= g_1\phi_{1,0}\beta\text{, where } g_1\phi_{1,0} = g_0, \\ &= g_1\alpha\phi_{1,0}\text{ , by (6.5),} \end{aligned} \tag{7.3}$$

Equations (7.2) and (7.3) imply that

$$\bar{\alpha}\psi = \bar{\beta} \tag{7.4}$$

where $\bar{\beta} = \beta|_{\overline{G_0}}$, and we can write (6.5) as

$$\alpha\phi_{1,0} = \phi_{1,0}\bar{\beta} \tag{7.5}$$

$$= \phi_{1,0}\bar{\alpha}\psi \tag{7.6}$$

Thus given $\alpha \in \mathrm{End}(G_1)$ such that $\mathrm{K}\alpha \subseteq \mathrm{K}$ and $\bar{\alpha}\psi = \bar{\beta}$ for some $\beta \in \mathrm{End}(G_0)$ satisfying (7.1), we can define a map

$$(\alpha, \beta)^{(1)} : S \longrightarrow S$$

$$(s)(\alpha,\beta)^{(1)} = \begin{cases} s\alpha & \text{if } s \in G_1\ , \\ s\beta & \text{if } s \in G_0\ , \text{ where } \bar{\beta} = \bar{\alpha}\psi\ . \end{cases}$$

The map $(\alpha, \beta)^{(1)}$ satisfies the relation, for all $s_1 \in G_1$

$$s_1\phi_{1,0}(\alpha,\beta)^{(1)} = s_1(\alpha,\beta)^{(1)}\phi_{1,0}, \tag{7.7}$$

since the left hand side of (7.7) gives

$$\begin{aligned} s_1\phi_{1,0}(\alpha,\beta)^{(1)} &= s_1\phi_{1,0}\beta \\ &= s_1\alpha\phi_{1,0}\text{, by (7.5),} \\ &= s_1(\alpha,\beta)^{(1)}\phi_{1,0}. \end{aligned}$$

Furthermore the map $(\alpha, \beta)^{(1)}$ is an endomorphism of type I on S. To see this, it is sufficient to consider the case when $s_1 \in G_1$ and $s_2 \in G_0$ (similarly when $s_1 \in G_0$ and $s_2 \in G_1$), so we have

$$(s_1s_2)(\alpha,\beta)^{(1)} = (s_0s_2)(\alpha,\beta)^{(1)}\text{, where } s_0 = s_1\phi_{1,0},$$

$$
\begin{aligned}
&= (s_0 s_2)\beta \\
&= (s_0\beta)(s_2\beta) \\
&= s_0(\alpha,\beta)^{(1)} s_2(\alpha,\beta)^{(1)} \\
&= s_1\phi_{1,0}(\alpha,\beta)^{(1)} s_2(\alpha,\beta)^{(1)} \\
&= s_1(\alpha,\beta)^{(1)}\phi_{1,0} s_2(\alpha,\beta)^{(1)}, \text{ by (7.7)}, \\
&= s_1(\alpha,\beta)^{(1)} s_2(\alpha,\beta)^{(1)}.
\end{aligned}
$$

Now consider the endomorphisms of type II on S. For such an endomorphism f, we have

$f|_{G_1} = \alpha$, say, where $\alpha \in \mathrm{End}(G_1)$ and $f|_{G_0} = \gamma$, say, where $\gamma \in \mathrm{Hom}(G_0, G_1)$.

We notice that proposition 6.2.1 can also be applied in this case and equation (6.11) may be written as

$$\overline{(\phi_{1,0}\gamma)}\psi = \overline{\gamma\phi_{1,0}} \tag{7.8}$$

where $\overline{\gamma\phi_{1,0}} \in \mathrm{End}(\overline{G_0})$.

Let $\overline{G_0}\gamma = \overline{G_1}$, where $\overline{G_1}$ is not necessarily the whole of G_1, then equation (6.8) is still valid which forces α to satisfy

$$G_1\alpha \subseteq \overline{G_1} \tag{7.9}$$

However, given $\gamma \in \mathrm{Hom}(G_0, G_1)$, we can define a map $\gamma^{(2)}$ in the same way as in chapter 6 as

$$\gamma^{(2)} : S \longrightarrow S$$

$$(s)\gamma^{(2)} = \begin{cases} s\phi_{1,0}\gamma & \text{if } s \in G_1 , \\ s\gamma & \text{if } s \in G_0. \end{cases}$$

As we have seen in chapter 6, the map $\gamma^{(2)}$ is an endomorphism of type II on S.

Finally we consider the endomorphisms of type III on S.

For such an endomorphism f, we have

$f|_{G_1} \in \mathrm{Hom}(G_1, G_0)$ and $f|_{G_0} \in \mathrm{End}(G_0)$.

Equation (6.9) is also satisfied and for a given β, an endomorphism of G_0, we can easily define a map $\beta^{(3)}$, as an endomorphism of type III on S, where

$$\beta^{(3)} : S \longrightarrow S$$

is given by

$$(s)\beta^{(3)} = \begin{cases} s\phi_{1,0}\beta & \text{if } s \in G_1 , \\ s\beta & \text{if } s \in G_0. \end{cases}$$

Now for $\alpha \in \mathrm{End}(G_1)$ and $\beta \in \mathrm{End}(G_0)$ satisfying $\mathrm{K}\alpha \subseteq \mathrm{K}$ and $\overline{G_0}\beta \subseteq \overline{G_0}$, we define

$$\mathrm{End}_{K,\phi}(G_1) := \{\alpha \in \mathrm{End}(G_1); \bar{\alpha}\psi = \bar{\beta} \text{ for some } \beta \in \mathrm{End}(G_0)\ \}$$

$$\mathrm{End}_{\phi}(G_0) := \{\beta \in \mathrm{End}(G_0); \bar{\beta} = \bar{\alpha}\psi \text{ for some } \alpha \in \mathrm{End}(G_1)\ \}.$$

So there is a correspondence (not necessarily 1-1) between the above two sets.

Also we define

$$\overline{\mathrm{End}(\overline{G_0})} := \{\nu \in \mathrm{End}(\overline{G_0}); \exists\, \alpha \in \mathrm{End}_{K,\phi}(G_1) \text{ such that } \bar{\alpha}\psi = \nu \text{ , and } \exists\, \beta \in \mathrm{End}_{\phi}(G_0) \text{ such that } \bar{\beta} = \nu\}.$$

Then there exist epimorphisms τ_1 and τ_2 such that

$$\tau_1 : \mathrm{End}_{K,\phi}(G_1) \longrightarrow \overline{\mathrm{End}(\overline{G_0})}$$

is defined by

$$(\alpha)\tau_1 = \bar{\alpha}\psi$$

and

$$\tau_2 : \mathrm{End}_{\phi}(G_0) \longrightarrow \overline{\mathrm{End}(\overline{G_0})}$$

is defined by

$$(\beta)\tau_2 = \bar{\beta}.$$

Let

E_1 : denote the set $\mathrm{End}_{K,\phi}(G_1)$

E_0 : denote the set $\mathrm{End}_{\phi}(G_0)$

$\overline{\mathrm{E}}$: denote the set $\overline{\mathrm{End}(\overline{G_0})}$.

Then we have the near-rings F_1 , F_0 and $\overline{\mathrm{F}}$, where

$$\begin{aligned} \mathrm{F}_1 &= \mathrm{Nr}\langle \mathrm{E}_1 \rangle \subseteq \mathrm{E}(G_1) \\ \mathrm{F}_0 &= \mathrm{Nr}\langle \mathrm{E}_0 \rangle \subseteq \mathrm{E}(G_0) \\ \overline{\mathrm{F}} &= \mathrm{Nr}\langle \overline{\mathrm{E}} \rangle \subseteq \mathrm{E}(\overline{G_0}). \end{aligned}$$

From above, we already have the homomorphisms

$$\tau_1 : \mathrm{E}_1 \longrightarrow \overline{\mathrm{E}}$$

$$\tau_2 : \mathrm{E}_0 \longrightarrow \overline{\mathrm{E}}.$$

Next we extend τ_1 and τ_2 to $\tau_1^\star$ and $\tau_2^\star$ respectively, where

$$\tau_1^\star : \mathrm{F}_1 \longrightarrow \overline{\mathrm{F}}$$

is given by

$$(\sum \epsilon_i\alpha_i)\tau_1^\star = \sum \epsilon_i(\bar{\alpha}_i\psi)$$

where $\epsilon_i = \pm 1$, for all i, and

$$\tau_2^\star : \mathrm{F}_0 \longrightarrow \overline{\mathrm{F}}$$

is given by

$$(\sum \eta_i\beta_j)\tau_2^\star = \sum \eta_j\bar{\beta}_j$$

where $\eta_j = \pm 1$, for all j.

First we show that $\tau_1^\star$ is a homomorphism. For this purpose we show that

$$(c)\tau_1^\star = 0 \text{ whenever } 0 = c \in \mathrm{F}_1.$$

Suppose that $c = 0 = \sum_{i=1}^{n} \epsilon_i\alpha_i \in \mathrm{F}_1$, then for $g_1 \in G_1$, we have

$$g_1 \sum_{i=1}^{n} \epsilon_i\alpha_i \;=\; 0.$$

Thus for $g_0 \in \overline{G_0}$, we have

$$\begin{aligned} g_0(c)\tau_1^\star &= g_0(\sum_{i=1}^{n} \epsilon_i\alpha_i)\tau_1^\star \\ &= g_0(\sum_{i=1}^{n} \epsilon_i\bar{\alpha}_i\psi) \\ &= g_1\phi_{1,0}(\sum_{i=1}^{n} \epsilon_i\bar{\alpha}_i\psi) \text{ , where } g_1\phi_{1,0} = g_0 \text{ ,} \\ &= g_1(\sum_{i=1}^{n} \epsilon_i\phi_{1,0}\bar{\alpha}_i\psi) \\ &= g_1(\sum_{i=1}^{n} \epsilon_i\alpha_i\phi_{1,0}) \text{ , by (7.6) ,} \\ &= 0. \end{aligned}$$

Hence $\tau_1^\star$ is a homomorphism.

That the map $\tau_2^\star$ is a homomorphism is obvious, since if $0 = d = \sum_{j=1}^m \eta_j\beta_j \in \mathrm{F}_0$, then $g_0(\sum_{j=1}^m \eta_j\beta_j) = 0$ for all $g_0 \in G_0$, and so for any $g_0 \in \overline{G_0} \subseteq G_0$, we have $g_0(d)\tau_2^\star = g_0(\sum_{j=1}^m \eta_j\bar{\beta_j}) = 0$, which shows that $\tau_2^\star$ is a homomorphism.

Let

$$\mathrm{P} := \{(\alpha, \beta) \subseteq \mathrm{E}_1 \times \mathrm{E}_0; \alpha\tau_1 = \beta\tau_2\}.$$

It can be seen easily that the set P is closed under product as follows; first we notice that for $\bar{\alpha_1}, \bar{\alpha_2} \in \mathrm{End}(G_1/\mathrm{K})$, we have

$$(\overline{\alpha_1\alpha_2})\psi \;=\; (\bar{\alpha_1}\psi)(\bar{\alpha_2}\psi).$$

Let $x, y \in \mathrm{P}$, where $x = (\alpha_1, \beta_1)$ and $y = (\alpha_2, \beta_2)$, then $xy = (\alpha_1\alpha_2, \beta_1\beta_2)$.

By definition of P, we have $\alpha_1\tau_1 = \beta_1\tau_2$ and $\alpha_2\tau_1 = \beta_2\tau_2$ which means that $\bar{\alpha_1}\psi = \bar{\beta_1}$ and $\bar{\alpha_2}\psi = \bar{\beta_2}$.

Thus

$$(\overline{\alpha_1\alpha_2})\psi = (\bar{\alpha_1}\psi)(\bar{\alpha_2}\psi) = \bar{\beta_1}\bar{\beta_2} = \overline{\beta_1\beta_2}$$

so the product xy satisfies

$$(\alpha_1\alpha_2)\tau_1 = (\beta_1\beta_2)\tau_2$$

Hence $xy \in \mathrm{P}$.

Let us define a map

$$\Gamma : \mathrm{P} \longrightarrow \mathrm{End}(S)$$

by

$$(\alpha, \beta)\Gamma = (\alpha, \beta)^{(1)}$$

We show that Γ is a homomorphism. Let $x, y \in \mathrm{P}$, where $x = (\alpha_1, \beta_1), y = (\alpha_2, \beta_2)$. Then

$$\begin{aligned}(xy)\Gamma &= ((\alpha_1, \beta_1)(\alpha_2, \beta_2))\Gamma \\ &= (\alpha_1\alpha_2, \beta_1\beta_2)\Gamma \\ &= (\alpha_1\alpha_2, \beta_1\beta_2)^{(1)}.\end{aligned}$$

So for $s_1 \in G_1$, we have

$$
\begin{aligned}
s_1(xy)\Gamma &= s_1(\alpha_1\alpha_2, \beta_1\beta_2)^{(1)} \\
&= s_1(\alpha_1\alpha_2)
\end{aligned}
\tag{7.10}
$$

and for $s_0 \in G_0$, we have

$$
\begin{aligned}
s_0(xy)\Gamma &= s_0(\alpha_1\alpha_2, \beta_1\beta_2)^{(1)} \\
&= s_0(\beta_1\beta_2).
\end{aligned}
\tag{7.11}
$$

On the other hand, for $s_1 \in G_1$,we have

$$
\begin{aligned}
s_1((x)\Gamma(y)\Gamma) &= s_1((\alpha_1, \beta_1)\Gamma(\alpha_2, \beta_2)\Gamma) \\
&= s_1((\alpha_1, \beta_1)^{(1)}(\alpha_2, \beta_2)^{(1)}) \\
&= (s_1\alpha_1)(\alpha_2, \beta_2)^{(1)} \\
&= s_1\alpha_1\alpha_2
\end{aligned}
\tag{7.12}
$$

and for $s_0 \in G_0$, we have

$$
\begin{aligned}
s_0((x)\Gamma(y)\Gamma) &= s_0((\alpha_1, \beta_1)\Gamma(\alpha_2, \beta_2)\Gamma) \\
&= s_0((\alpha_1, \beta_1)^{(1)}(\alpha_2, \beta_2)^{(1)}) \\
&= (s_0\beta_1)(\alpha_2, \beta_2)^{(1)} \\
&= s_0\beta_1\beta_2
\end{aligned}
\tag{7.13}
$$

Equations (7.10) – (7.13) show that $(xy)\Gamma = (x)\Gamma(y)\Gamma$ and Γ is a homomorphism. Let

$$\overline{\mathrm{P}} = \mathrm{Snr}\langle\mathrm{P}\rangle \subseteq \mathrm{F}_1 \times \mathrm{F}_0.$$

Then we may extend the map Γ to $\Gamma^\star$ such that

$$\Gamma^\star : \overline{\mathrm{P}} \longrightarrow E(S)$$

is defined by

$$(a, b)\Gamma^\star = \textstyle\sum_{i=1}^{n} \epsilon_i(\alpha_i, \beta_i)^{(1)}$$

where $a = \sum_{i=1}^{n} \epsilon_i \alpha_i, b = \sum_{i=1}^{n} \epsilon_i \beta_i, \epsilon_i = \pm 1$ and each endomorphism $(\alpha_i, \beta_i)^{(1)}$ corresponds to the pair α_i, β_i in the way they arise in the definition of $(\alpha, \beta)^{(1)}$.

The above map, $\Gamma^\star$, is a homomorphism, for if we suppose that $0 = (a, b) \in \overline{\mathrm{P}}$, where $a = \sum_{i=1}^{n} \epsilon_i \alpha_i$ and $b = \sum_{i=1}^{n} \epsilon_i \beta_i$, then

$$g_1 \sum_{i=1}^{n} \epsilon_i \alpha_i = 0 \text{ and } g_0 \sum_{i=1}^{n} \epsilon_i \beta_i = 0, \text{for all } g_1 \in G_1, g_0 \in G_0.$$

Thus, $g_1 \sum_{i=1}^{n} \epsilon_i (\alpha_i, \beta_i)^{(1)} = 0$ and $g_0 \sum_{i=1}^{n} \epsilon_i (\alpha_i, \beta_i)^{(1)} = 0$ for all $g_1 \in G_1, g_0 \in G_0$.

Hence $\sum_{i=1}^{n} \epsilon_i (\alpha_i, \beta_i)^{(1)} = 0$, which shows that $(a, b)\Gamma^\star = 0$ and $\Gamma^\star$ is a homomorphism.

Moreover, $\Gamma^\star$ is a monomorphism. To see that let us suppose that $0 \neq (a, b) \in \overline{\mathrm{P}}$, where $a = \sum_{i=1}^{n} \epsilon_i \alpha_i$ and $b = \sum_{i=1}^{n} \epsilon_i \beta_i$; then either $a \neq 0$ or $b \neq 0$.

If $a \neq 0$, then there exists $g_1 \in G_1$ such that $g_1 \sum_{i=1}^{n} \epsilon_i \alpha_i \neq 0$, while for each α_i there exists $\beta_i \in \mathrm{End}(G_0)$ such that $\bar{\alpha}_i \psi = \beta_i$. It follows that

$$0 \neq g_1 \sum_{i=1}^{n} \epsilon_i (\alpha_i, \beta_i)^{(1)} = g_1 (a, b)\Gamma^\star.$$

If $b \neq 0$, then there exists $g_0 \in G_0$ such that $g_0 \sum_{i=1}^{n} \epsilon_i \beta_i \neq 0$, while for each β_i there exists $\alpha_i \in \mathrm{End}(G_1)$ such that $\bar{\beta}_i = \bar{\alpha}_i \psi$. So again we have

$$0 \neq g_0 \sum_{i=1}^{n} \epsilon_i (\alpha_i, \beta_i)^{(1)} = g_0 (a, b)\Gamma^\star.$$

Hence $(a, b)\Gamma^\star \neq 0$ and $\Gamma^\star$ is a monomorphism.

Thus we have shown that the endomorphisms of type I on S will generate in $E(S)$ a subnear-ring, which we will denote by $E_{K,\phi}(G_1, \overline{G_0})^I$, isomorphic to $\overline{\mathrm{P}}$, where $\overline{\mathrm{P}}$ is a subnear-ring of $\mathrm{F}_1 \oplus \mathrm{F}_0$. It can be seen that the maps π_1 and π_0 are onto, where

$$\pi_1 : \overline{\mathrm{P}} \longrightarrow \mathrm{F}_1$$

is defined by

$$(a, b)\pi_1 = a$$

and

$$\pi_0 : \overline{\mathrm{P}} \longrightarrow \mathrm{F}_0$$

is defined by

$$(a,b)\pi_0 = b$$

Hence, $E_{K,\phi}(G_1,\overline{G_0})^I$ is a subdirect product of $\mathrm{F}_1 \oplus \mathrm{F}_0$.

Next we consider the set $\mathrm{Hom}(G_0, G_1)$ and follow the same pattern as in chapter 6; then we consider the group $(\mathrm{gp}< \mathrm{Hom}(G_0,G_1) >, +)$ and obtain an isomorphic copy of this group inside $E(S)$. This covers the endomorphisms of type II.

Finally we consider the endomorphisms of type III on S and proceed along the same way as in chapter 6 to see that the endomorphisms of type III on S will generate in $E(S)$ a subnear-ring isomorphic to the near-ring $E(G_0)$.

Hence, a similar situation to (6.10), we now have

$$E(S) \;=\; E_{K,\phi}(G_1,\overline{G_0})^I \cup E(G_0,G_1)^{II} \cup E(G_0)^{III} \tag{7.14}$$

where $E_{K,\phi}(G_1,\overline{G_0})^I$ is the copy of the near-ring $\overline{\mathrm{P}} \subseteq \mathrm{F}_1 \oplus \mathrm{F}_0$ in $E(S)$ which is generated by the endomorphisms of type I on S, and $E(G_0,G_1)^{II}$ is an isomorphic copy of the group $(\mathrm{gp}< \mathrm{Hom}(G_0,G_1) >, +)$ in $E(S)$, while $E(G_0)^{III}$ is the copy of the near-ring $E(G_0)$ which is generated by the endomorphisms of type III.

The next step is to look at the sum inside $E(S)$.

7.2 Addition in $E(S)$

We start by studying the sum of endomorphisms of different types, then we proceed to describe the sum in $E(S)$ with regard to equation (7.14).

Let us first consider the sum of endomorphisms of type I with those of type II. suppose that $(\alpha,\beta)^{(1)}$ and $\gamma^{(2)}$ are endomorphisms of type I and type II respectively. For $g_1 \in G_1$, we have

$$\begin{aligned} g_1((\alpha,\beta)^{(1)} + \gamma^{(2)}) &= g_1(\alpha,\beta)^{(1)} + g_1\gamma^{(2)} \\ &= g_1(\alpha,\beta)^{(1)} + g_1\phi_{1,0}\gamma \end{aligned}$$

but equation (7.8) allows us to write the right hand side of the above equality as

$$\begin{aligned} &= g_1(\alpha,\beta)^{(1)} + g_1(\phi_{1,0}\gamma,\gamma\phi_{1,0})^{(1)} \\ &= g_1((\alpha,\beta)^{(1)} + (\phi_{1,0}\gamma,\gamma\phi_{1,0})^{(1)}); \end{aligned} \tag{7.15}$$

also for $g_0 \in G_0$, we have

$$\begin{aligned} g_0((\alpha,\beta)^{(1)} + \gamma^{(2)}) &= g_0(\alpha,\beta)^{(1)} + g_0\gamma^{(2)} \\ &= g_0\beta + g_0\gamma \\ &= g_0\beta + g_0\gamma\phi_{1,0} \\ &= g_0\beta + g_0(\phi_{1,0}\gamma,\gamma\phi_{1,0})^{(1)} \text{ , with the help of equation (7.8),} \\ &= g_0(\alpha,\beta)^{(1)} + g_0(\phi_{1,0}\gamma,\gamma\phi_{1,0})^{(1)} \\ &= g_0((\alpha,\beta)^{(1)} + (\phi_{1,0}\gamma,\gamma\phi_{1,0})^{(1)}). \end{aligned} \tag{7.16}$$

Equations (7.15) and (7.16) imply that

$$(\alpha,\beta)^{(1)} + \gamma^{(2)} = (\alpha,\beta)^{(1)} + (\phi_{1,0}\gamma,\gamma\phi_{1,0})^{(1)}. \tag{7.17}$$

Similarly we can show that

$$\gamma^{(2)} + (\alpha,\beta)^{(1)} = (\phi_{1,0}\gamma,\gamma\phi_{1,0})^{(1)} + (\alpha,\beta)^{(1)}. \tag{7.18}$$

Hence the sum of an endomorphism of type I with an endomorphism of type II could be considered as a sum of type I.

Next we consider the sum $(\alpha,\beta)^{(1)} + \nu^{(3)}$ of endomorphisms of type I and type III respectively. For $g_1 \in G_1$, we have

$$\begin{aligned} g_1((\alpha,\beta)^{(1)} + \nu^{(3)}) &= g_1(\alpha,\beta)^{(1)} + g_1\nu^{(3)} \\ &= g_1\alpha + g_1\phi_{1,0}\nu \\ &= g_1\alpha\phi_{1,0} + g_1\phi_{1,0}\nu \\ &= g_1\phi_{1,0}\bar{\beta} + g_1\phi_{1,0}\nu \text{ , by (7.5) ,} \\ &= g_1\beta^{(3)} + g_1\nu^{(3)} \\ &= g_1(\beta^{(3)} + \nu^{(3)}), \end{aligned} \tag{7.19}$$

and for $g_0 \in G_0$, we have

$$\begin{aligned} g_0((\alpha,\beta)^{(1)} + \nu^{(3)}) &= g_0(\alpha,\beta)^{(1)} + g_0\nu^{(3)} \\ &= g_0\beta + g_0\nu^{(3)} \\ &= g_0\beta^{(3)} + g_0\nu^{(3)} \\ &= g_0(\beta^{(3)} + \nu^{(3)}). \end{aligned} \tag{7.20}$$

Equations (7.19) and (7.20) imply that

$$(\alpha,\beta)^{(1)} + \nu^{(3)} = \beta^{(3)} + \nu^{(3)}. \tag{7.21}$$

Similarly we can get

$$\nu^{(3)} + (\alpha,\beta)^{(1)} = \nu^{(3)} + \beta^{(3)} \tag{7.22}$$

which shows that the sum of two endomorphisms of type I and type III could be assumed to be a sum of endomorphisms of type III.

Finally we consider the sum $\gamma^{(2)} + \nu^{(3)}$ of endomorphisms of type II and type III respectively. For $g_1 \in G_1$, we have

$$\begin{aligned} g_1(\gamma^{(2)} + \nu^{(3)}) &= g_1\gamma^{(2)} + g_1\nu^{(3)} \\ &= g_1\phi_{1,0}\gamma + g_1\phi_{1,0}\nu \\ &= g_1\phi_{1,0}\gamma\phi_{1,0} + g_1\phi_{1,0}\nu \\ &= g_1\phi_{1,0}(\gamma\phi_{1,0}) + g_1\nu^{(3)} \\ &= g_1(\gamma\phi_{1,0})^{(3)} + g_1\nu^{(3)} \\ &= g_1((\gamma\phi_{1,0})^{(3)} + \nu^{(3)}), \end{aligned} \tag{7.23}$$

and for $g_0 \in G_0$, we have

$$\begin{aligned} g_0(\gamma^{(2)} + \nu^{(3)}) &= g_0\gamma^{(2)} + g_0\nu^{(3)} \\ &= g_0\gamma + g_0\nu \\ &= g_0\gamma\phi_{1,0} + g_0\nu \\ &= g_0(\gamma\phi_{1,0})^{(3)} + g_0\nu^{(3)} \\ &= g_0((\gamma\phi_{1,0})^{(3)} + \nu^{(3)}). \end{aligned} \tag{7.24}$$

Equations (7.23) and (7.24) imply that

$$\gamma^{(2)} + \nu^{(3)} = (\gamma\phi_{1,0})^{(3)} + \nu^{(3)}. \tag{7.25}$$

Similarly we can get

$$\nu^{(3)} + \gamma^{(2)} = \nu^{(3)} + (\gamma\phi_{1,0})^{(3)}. \tag{7.26}$$

Hence the sum of two endomorphisms of type II and III could be considered as a sum of endomorphisms of type III.

By induction, from equations (7.17), (7.21) and (7.25) we can obtain the following

$$\sum_{i=1}^{n} \epsilon_i(\alpha_i, \beta_i)^{(1)} + \sum_{j=1}^{k} \eta_j \gamma_j^{(2)} = \sum_{i=1}^{n} \epsilon_i(\alpha_i, \beta_i)^{(1)} + \sum_{j=1}^{k} \eta_j(\phi_{1,0}\gamma_j, \gamma_j\phi_{1,0})^{(1)}, \tag{7.27}$$

$$\sum_{i=1}^{n} \epsilon_i(\alpha_i, \beta_i)^{(1)} + \sum_{r=1}^{m} \delta_r \nu_r^{(3)} = \sum_{i=1}^{n} \epsilon_i \beta_i^3 + \sum_{r=1}^{m} \delta_r \nu_r^{(3)}, \tag{7.28}$$

$$\sum_{j=1}^{k} \eta_j \gamma_j^{(2)} + \sum_{r=1}^{m} \delta_r \nu_r^{(3)} = \sum_{j=1}^{k} \eta_j (\gamma_j \phi_{1,0})^{(3)} + \sum_{r=1}^{m} \delta_r \nu_r^{(3)} \tag{7.29}$$

where,

$(\ )^{(1)} \in$ type I , $(\ )^{(2)} \in$ type II , $(\ )^{(3)} \in$ type III and $\epsilon_i = \pm 1$, $\eta_j = \pm 1$, $\delta_r = \pm 1$, $i = 1, \ldots, n$, $r = 1, \ldots, m$, $j = 1, \ldots, k$.

As mentioned earlier, we can obtain the same conclusion if we reverse the order of addition in the above equations. Hence, we can see that the sum in $E(S)$ will obey the following :

$E_{K,\phi}(G_1, \overline{G_0})^I + E(G_0, G_1)^{II} \longrightarrow E_{K,\phi}(G_1, \overline{G_0})^I$ and

$E(G_0, G_1)^{II} + E_{K,\phi}(G_1, \overline{G_0})^I \longrightarrow E_{K,\phi}(G_1, \overline{G_0})^I$,

$E_{K,\phi}(G_1, \overline{G_0})^I + E(G_0)^{III} \longrightarrow E(G_0)^{III}$ and

$E(G_0)^{III} + E_{K,\phi}(G_1, \overline{G_0})^I \longrightarrow E(G_0)^{III}$,

$E(G_0, G_1)^{II} + E(G_0)^{III} \longrightarrow E(G_0)^{III}$ and

$E(G_0)^{III} + E(G_0, G_1)^{II} \longrightarrow E(G_0)^{III}$.

Next we look at the product in $E(S)$.

7.3 Product in $E(S)$

We have seen in chapter 6 that the endomorphisms of each type are closed under product. The same thing can be said about the product here. So we check the product of endomorphisms of different types. Let $(\alpha,\beta)^{(1)},\gamma^{(2)}$ be endomorphisms of type I and type II respectively. Then for $g_1 \in G_1$, we have

$$\begin{aligned} g_1((\alpha,\beta)^{(1)}\gamma^{(2)}) &= (g_1\alpha)\gamma^{(2)} \\ &= g_1\alpha\phi_{1,0}\gamma \\ &= g_1\phi_{1,0}\beta\gamma\,,\text{ by (7.5)}\,, \\ &= g_1(\beta\gamma)^{(2)}, \end{aligned} \tag{7.30}$$

and for $g_0 \in G_0$, we have

$$\begin{aligned} g_0((\alpha,\beta)^{(1)}\gamma^{(2)}) &= (g_0\beta)\gamma^{(2)} \\ &= g_0\beta\gamma \\ &= g_0(\beta\gamma)^{(2)}. \end{aligned} \tag{7.31}$$

Equations (7.30) and (7.31) show that

$$(\alpha,\beta)^{(1)}\gamma^{(2)} = (\beta\gamma)^{(2)} \tag{7.32}$$

Reversing the order of the maps in the above product, we get for $g_1 \in G_1$,

$$\begin{aligned} g_1(\gamma^{(2)}(\alpha,\beta)^{(1)}) &= (g_1\phi_{1,0}\gamma)(\alpha,\beta)^{(1)} \\ &= g_1\phi_{1,0}\gamma\alpha \\ &= g_1(\gamma\alpha)^{(2)}, \end{aligned} \tag{7.33}$$

and for $g_0 \in G_0$, we have

$$\begin{aligned} g_0(\gamma^{(2)}(\alpha,\beta)^{(1)}) &= (g_0\gamma)(\alpha,\beta)^{(1)} \\ &= g_0\gamma\alpha \\ &= g_0(\gamma\alpha)^{(2)}. \end{aligned} \tag{7.34}$$

Equations (7.33) and (7.34) imply that

$$\gamma^{(2)}(\alpha,\beta)^{(1)} = (\gamma\alpha)^{(2)}. \tag{7.35}$$

Next we consider the product of $(\alpha,\beta)^{(1)}$ with $\nu^{(3)}$. For $g_1 \in G_1$, we have

$$\begin{aligned} g_1((\alpha,\beta)^{(1)}\nu^{(3)}) &= (g_1\alpha)\nu^{(3)} \\ &= g_1\alpha\phi_{1,0}\nu \\ &= g_1\phi_{1,0}\beta\nu, \text{by } (7.5)\ , \\ &= g_1(\beta\nu)^{(3)}, \end{aligned} \tag{7.36}$$

and for $g_0 \in G_0$, we have

$$\begin{aligned} g_0((\alpha,\beta)^{(1)}\nu^{(3)}) &= (g_0\beta)\nu^{(3)} \\ &= g_0\beta\nu \\ &= g_0(\beta\nu)^{(3)}. \end{aligned} \tag{7.37}$$

Equations (7.36) and (7.37) imply that

$$(\alpha,\beta)^{(1)}\nu^{(3)} = (\beta\nu)^{(3)}. \tag{7.38}$$

Reversing the order of the maps in the above product, we get for $g_1 \in G_1$,

$$\begin{aligned} g_1(\nu^{(3)}(\alpha,\beta)^{(1)}) &= (g_1\phi_{1,0}\nu)(\alpha,\beta)^{(1)} \\ &= g_1\phi_{1,0}\nu\beta \\ &= g_1(\nu\beta)^{(3)}, \end{aligned} \tag{7.39}$$

and for $g_0 \in G_0$, we have

$$\begin{aligned} g_0(\nu^{(3)}(\alpha,\beta)^{(1)}) &= (g_0\nu)(\alpha,\beta)^{(1)} \\ &= g_0\nu\beta \\ &= g_0(\nu\beta)^{(3)}. \end{aligned} \tag{7.40}$$

Equations (7.39) and (7.40) give that

$$\nu^{(3)}(\alpha,\beta)^{(1)} = (\nu\beta)^{(3)}. \tag{7.41}$$

Finally we consider the product of $\gamma^{(2)}$ with $\nu^{(3)}$. For $g_1 \in G_1$, we have

$$\begin{aligned} g_1(\gamma^{(2)}\nu^{(3)}) &= (g_1\phi_{1,0}\gamma)\nu^{(3)} \\ &= g_1\phi_{1,0}\gamma\phi_{1,0}\nu \\ &= g_1(\gamma\phi_{1,0}\nu)^{(3)}, \end{aligned} \tag{7.42}$$

and for $g_0 \in G_0$, we have

$$\begin{aligned} g_0(\gamma^{(2)}\nu^{(3)}) &= (g_0\gamma)\nu^{(3)} \\ &= g_0\gamma\phi_{1,0}\nu \\ &= g_0(\gamma\phi_{1,0}\nu)^{(3)}. \end{aligned} \tag{7.43}$$

Equations (7.42) and (7.43) imply that

$$\gamma^{(2)}\nu^{(3)} = (\gamma\phi_{1,0}\nu)^{(3)}. \tag{7.44}$$

Now we consider the above product when reversing the order of the maps to get, for $g_1 \in G_1$,

$$\begin{aligned} g_1(\nu^{(3)}\gamma^{(2)}) &= (g_1\phi_{1,0}\nu)\gamma^{(2)} \\ &= g_1\phi_{1,0}\nu\gamma \\ &= g_1(\nu\gamma)^{(2)}, \end{aligned} \tag{7.45}$$

and for $g_0 \in G_0$, we have

$$\begin{aligned} g_0(\nu^{(3)}\gamma^{(2)}) &= (g_0\nu)\gamma^{(2)} \\ &= g_0\nu\gamma \\ &= g_0(\nu\gamma)^{(2)}. \end{aligned} \tag{7.46}$$

Equations (7.45) and (7.46) imply that

$$\nu^{(3)}\gamma^{(2)} = (\nu\gamma)^{(2)}, \tag{7.47}$$

which gives a different result from equation (7.44). So the product in $E(S)$ can be represented as follows

$E_{K,\phi}(G_1, \overline{G_0})^I.E(G_0, G_1)^{II} \longrightarrow E(G_0, G_1)^{II}$ and
$E(G_0, G_1)^{II}.E_{K,\phi}(G_1, \overline{G_0})^I \longrightarrow E(G_0, G_1)^{II}$,

$E_{K,\phi}(G_1, \overline{G_0})^I.E(G_0)^{III} \longrightarrow E(G_0)^{III}$ and
$E(G_0)^{III}.E_{K,\phi}(G_1, \overline{G_0})^I \longrightarrow E(G_0)^{III}$,

$E(G_0, G_1)^{II}.E(G_0)^{III} \longrightarrow E(G_0)^{III}$ but
$E(G_0)^{III}.E(G_0, G_1)^{II} \longrightarrow E(G_0, G_1)^{II}$

Thus the product in $E(S)$ fails to yield a semilattice of semigroups.

Now we go back to addition in $E(S)$ to get our result.

7.4 Conclusion III

Let

$$A = E_{K,\phi}(G_1, \overline{G_0})^I,\ B = E(G_0, G_1)^{II},\ C = E(G_0)^{III},$$

then relation (7.14) can be written as

$$E(S) \;=\; A \cup B \cup C. \tag{7.48}$$

Let a, b and c be the zero maps in A, B and C respectively; then we have the semilattice

$$\mathcal{L} = \{a, b, c\}, \text{ with } c \leq a \leq b.$$

Define the following linking homomorphisms $\phi_{b,a}, \phi_{b,c}$ and $\phi_{a,c}$ such that

$$\phi_{b,a} : B \longrightarrow A$$
$$\textstyle\sum_j \eta_j \gamma_j^{(2)} \longrightarrow \sum_j \eta_j (\phi_{1,0}\gamma_j, \gamma_j \phi_{1,0})^{(1)},$$

$$\phi_{b,c} : B \longrightarrow C$$
$$\textstyle\sum_j \eta_j \gamma_j^{(2)} \longrightarrow \sum_j \eta_j (\gamma_j \phi_{1,0})^{(3)},$$

$$\phi_{a,c} : A \longrightarrow C$$
$$\textstyle\sum_i \epsilon_i (\alpha_i, \beta_i)^{(1)} \longrightarrow \sum_i \epsilon_i \beta_i^{(3)}.$$

Hence we have a strong semilattice of groups given by

$$E(S) = (\mathcal{L}, \{A, B, C\}, \{\phi_{b,a}, \phi_{b,c}, \phi_{a,c}\})$$

that is,

$(E(S), +)$ is a Clifford semigroup.

Chapter 8

Seminear-ring of endomorphisms IV

8.1 Starting case IV

Our purpose in this chapter is to generalize case I in chapter 5 in another direction.

Let

$$Y = \{0, 1, \ldots, n\}$$

be a semilattice with $n \geq \ldots \geq 1 \geq 0$.

Let $\{G_i\}_{i \in Y}$ be a family of isomorphic groups. We consider a group G such that

$$G \cong_{\theta_i} G_i, i = 0, 1, \ldots, n.$$

Consider $S = \bigcup_{i \in Y} G_i$. For each pair of groups G_i, G_j such that $i \geq j$, let

$$\phi_{i,j} : G_i \longrightarrow G_j$$

be the isomorphism mapping G_i to G_j. Then for $0 \leq j \leq i \leq n$, the isomorphism θ_j can be defined by

$$\theta_j = \theta_i \phi_{i,j} \tag{8.1}$$

Let $f \in \mathrm{End}(S)$, then by theorem 4.2.3, $f|_Y$ is an endomorphism of Y. We need to know more about the endomorphisms of Y in order to proceed to construct types of endomorphisms on S. However, the endomorphisms of Y cannot be found easily as in case I. Therfore, we first prove the following proposition which will give the spark to study the case.

Proposition 8.1.1 Suppose $W = \{1, \ldots, n\}$ is a semilattice with $1 \leq \ldots \leq n$. Then

$$| \operatorname{End}(W) | = \sum_{m=1}^{n} \binom{n}{m} \binom{n-1}{m-1}$$

Proof Let f be an endomorphism of the semilattice $W = \{1, \ldots, n\}$. Then we have a subset $X \subseteq W$ such that $Wf = X \subseteq W$. Suppose that $|X| = m$, then $1 \leq m \leq n$. So there are $\binom{n}{m}$ ways of choosing X. Now we must partition W into m subsets which are mapped into X. The number of ways we can do this is the number of ways in which we can put n identical objects into m non-empty sets; this having been done we number the objects 1 to n in sequence, map the first block to the first element of X, and so on. Now consider $n+m-m-1$ slots, place $m-1$ dividers to make m sections, and distribute the remaining $n-m$ objects in the slots, having assigned one object to each section already. This can be done in $\binom{n-1}{m-1}$ ways. But we already know that the number of ways of choosing X is $\binom{n}{m}$ ways. Hence

$$| \operatorname{End}(W) | = \sum_{m=1}^{n} \binom{n}{m} \binom{n-1}{m-1}.$$

Now if we think about the endomorphisms of Y, then a typical type of endomorphism of Y could be expressed in the form

$$\begin{pmatrix} k_1 & k_2 & \ldots & k_m \\ r_1 & r_2 & \ldots & r_m \end{pmatrix},$$

such that $0 \leq r_1 < r_2 < \ldots < r_m \leq n, \sum_{j=1}^{m} k_j = n+1\,, j = 1, \ldots, m$, where

r_1 is the image of the first k_1 elements of Y,

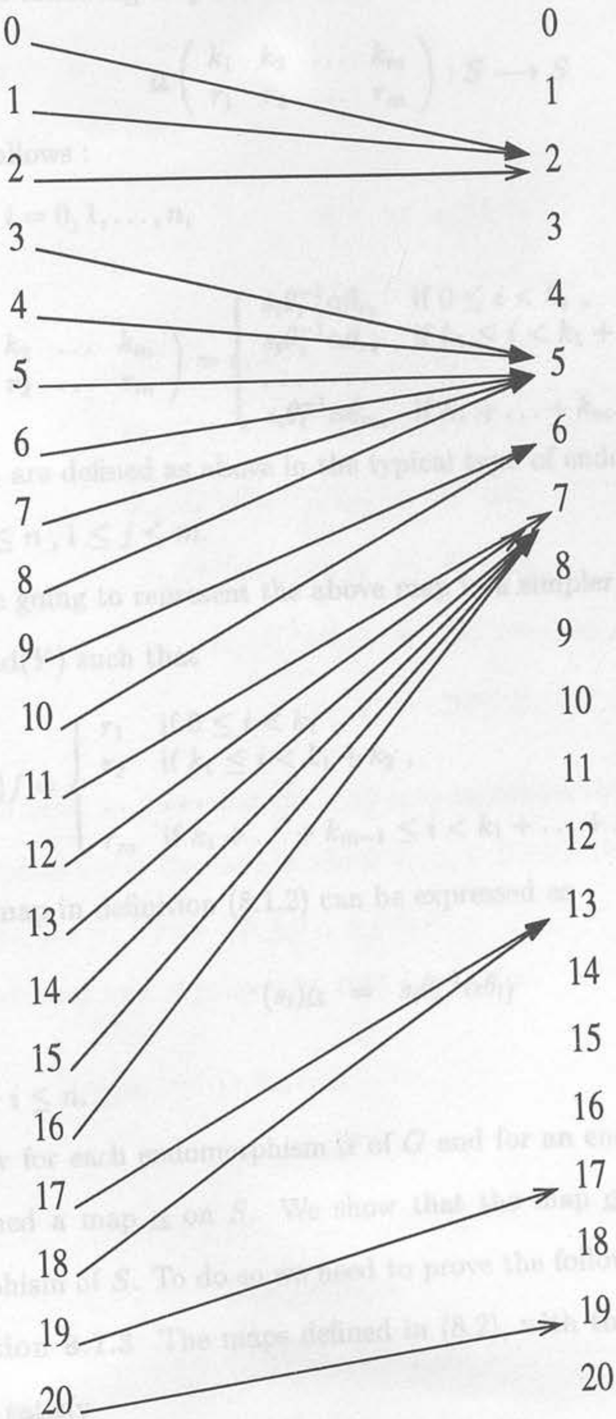
r_2 is the image of the next k_2 elements of Y,

. . . .

r_m is the image of the last k_m elements of Y.

For example, if $Y = \{0, 1, 2, \ldots, 20\}$. Then

type $\begin{pmatrix} 3 & 6 & 2 & 6 & 2 & 1 & 1 \\ 2 & 5 & 6 & 7 & 13 & 17 & 19 \end{pmatrix}$ will denote the following endomorphism :


0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20
0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20

Definition 8.1.2 Let $\alpha \in \mathrm{End}(G)$. For a typical type of endomorphism of Y, we define the following map on S

$$\underline{\alpha}\begin{pmatrix} k_1 & k_2 & \ldots & k_m \\ r_1 & r_2 & \ldots & r_m \end{pmatrix} : S \longrightarrow S$$

defined as follows :

for $s_i \in G_i$, $i = 0, 1, \ldots, n$,

$$(s_i)\underline{\alpha}\begin{pmatrix} k_1 & k_2 & \ldots & k_m \\ r_1 & r_2 & \ldots & r_m \end{pmatrix} = \begin{cases} s_i\theta_i^{-1}\alpha\theta_{r_1} & \text{if } 0 \le i < k_1 , \\ s_i\theta_i^{-1}\alpha\theta_{r_2} & \text{if } k_1 \le i < k_1 + k_2 , \\ \ldots\ldots\ldots & \\ s_i\theta_i^{-1}\alpha\theta_{r_m} & \text{if } k_1 + \ldots + k_{m-1} \le i < k_1 + \ldots + k_m = n+1 , \end{cases}$$

where k_j, r_j are defined as above in the typical type of endomorphism of Y, such that $0 \le i \le n$, $1 \le j \le m$.

Now we are going to represent the above map in a simpler way.

Let $f \in \mathrm{End}(Y)$ such that

$$(i)f = \begin{cases} r_1 & \text{if } 0 \le i < k_1 , \\ r_2 & \text{if } k_1 \le i < k_1 + k_2 , \\ \ldots\ldots & \\ r_m & \text{if } k_1 + \ldots + k_{m-1} \le i < k_1 + \ldots + k_m = n+1 . \end{cases}$$

Then the map in definition (8.1.2) can be expressed as

$$(s_i)\underline{\alpha} \;=\; s_i\theta_i^{-1}\alpha\theta_{if} \tag{8.2}$$

where $0 \le i \le n$.

So far now for each endomorphism α of G and for an endomorphism f of Y we have defined a map $\underline{\alpha}$ on S. We show that the map $\underline{\alpha}$ defined in (8.2) is an endomorphism of S. To do so we need to prove the following proposition.

Proposition 8.1.3 The maps defined in (8.2), with the same assumption and for $i \ge j$, satisfy

$$\phi_{i,j}\underline{\alpha} \;=\; \underline{\alpha}\phi_{if,jf}. \tag{8.3}$$

Proof Consider the isomorphisms

$$\phi_{i,j} : G_i \longrightarrow G_j$$

where $i \geq j$, and

$$\theta_k : G \longrightarrow G_k$$

where $k = 0, \ldots, n$.

Since $\theta_j = \theta_i \phi_{i,j}$, we have

$$\begin{aligned}
\theta_j^{-1} &= \phi_{i,j}^{-1}\theta_i^{-1} \\
\theta_j^{-1}\alpha\theta_{if} &= \phi_{i,j}^{-1}\theta_i^{-1}\alpha\theta_{if} \\
\phi_{i,j}\theta_j^{-1}\alpha\theta_{if} &= \theta_i^{-1}\alpha\theta_{if} \\
\phi_{i,j}\theta_j^{-1}\alpha(\theta_{jf}\theta_{jf}^{-1})\theta_{if} &= \theta_i^{-1}\alpha\theta_{if} \\
\phi_{i,j}(\theta_j^{-1}\alpha\theta_{jf})\theta_{jf}^{-1}\theta_{if} &= \theta_i^{-1}\alpha\theta_{if} \\
\phi_{i,j}(\theta_j^{-1}\alpha\theta_{jf})\phi_{if,jf}^{-1} &= \theta_i^{-1}\alpha\theta_{if} \\
\phi_{i,j}(\theta_j^{-1}\alpha\theta_{jf}) &= (\theta_i^{-1}\alpha\theta_{if})\phi_{if,jf} \\
\phi_{i,j}\underline{\alpha} &= \underline{\alpha}\phi_{if,jf}.
\end{aligned}$$

Now we are back to show that the map $\underline{\alpha}$ is an endomorphism of S. To do this we only need to verify the case in which $s_i, s_j \in S$ such that $s_i \in G_i, s_j \in G_j$ for some groups G_i, G_j where $i \neq j, 0 \leq i, j \leq n$.

Suppose, without loss of generality, that $i \geq j$, then

$$\begin{aligned}
(s_is_j)\underline{\alpha} &= (s_i\phi_{i,j}s_j)\underline{\alpha} \\
&= (s_i\phi_{i,j}s_j)(\theta_j^{-1}\alpha\theta_{jf}) \\
&= s_i\phi_{i,j}(\theta_j^{-1}\alpha\theta_{jf})s_j(\theta_j^{-1}\alpha\theta_{jf}) \\
&= (s_i\phi_{i,j})\underline{\alpha}(s_j)\underline{\alpha} \\
&= (s_i\underline{\alpha})\phi_{if,jf}(s_j)\underline{\alpha}\text{, by proposition (8.1.3),} \\
&= (s_i)\underline{\alpha}(s_j)\underline{\alpha}.
\end{aligned}$$

Hence $\underline{\alpha}$ is an endomorphism of S. This shows that an endomorphism of G can give rise to an endomorphism of the semigroup S. By proposition (8.1.1),

$|\mathrm{End}(Y)| = \Sigma_{m=1}^{n+1}\begin{pmatrix} n+1 \\ m \end{pmatrix}\begin{pmatrix} n \\ m-1 \end{pmatrix} = L$, say. Hence an endomorphism α of G will give rise to an endomorphism of S in L ways, as we have L types of endomorphisms of S. Now we are going to link up the endomorphisms of G with these types of endomorphisms of S.

Let

$$Q = \{1, 2, \ldots, L\}.$$

Then we can write

$$\mathrm{End}(Y) = \{f_q\, ; q \in Q\}.$$

For $\alpha \in \mathrm{End}(G)$ and $f_q \in \mathrm{End}(Y)$, we define $\underline{\alpha}_q$, an endomorphism of S, as follows

$$\underline{\alpha}_q : S \longrightarrow S$$

is given by

$$(s_i)\underline{\alpha}_q = s_i\theta_i^{-1}\alpha\theta_{if_q}, \tag{8.4}$$

for $s_i \in G_i \subseteq S, 0 \leq i \leq n$.

The endomorphism $\underline{\alpha}_q$ is called an endomorphism of type f_q on S.

For $q \in Q$, we define a map Γ_q where

$$\Gamma_q : \mathrm{End}(G) \longrightarrow \mathrm{End}(S)$$

is given by

$$(\alpha)\Gamma_q = \underline{\alpha}_q.$$

We show that the maps $\Gamma_q, q \in Q$, are one-to-one. Suppose that $\alpha_1, \alpha_2 \in \mathrm{End}(G)$ such that $\alpha_1 \neq \alpha_2$, then there exists $g \in G$ such that $g\alpha_1 \neq g\alpha_2$. But $g = g_i\theta_i^{-1}$ for some $g_i \in G_i$, where $i \in \{0, 1, \ldots, n\}$. Thus we have

$$\begin{aligned} g_i\theta_i^{-1}\alpha_1 &\neq g_i\theta_i^{-1}\alpha_2, \\ g_i\theta_i^{-1}\alpha_1\theta_{if_q} &\neq g_i\theta_i^{-1}\alpha_2\theta_{if_q}, \text{ as } \theta_{if_q} \text{ is 1-1}, \\ g_i\underline{\alpha_1}_q &\neq g_i\underline{\alpha_2}_q, \\ (\alpha_1)\Gamma_q &\neq (\alpha_2)\Gamma_q. \end{aligned}$$

Hence the maps $\Gamma_q, q \in Q$, are one-to-one maps.

Next we extend the maps $\Gamma_q, q \in Q$, to $\Gamma_q^\star, q \in Q$, such that

$$\Gamma_q^\star : E(G) \longrightarrow E(S)$$

are given by

$$(\sum \epsilon_r \alpha_r)\Gamma_q^\star = \sum \epsilon_r \underline{\alpha_r}_q$$

where $\epsilon_r = \pm 1$.

We show that the maps $\Gamma_q^\star, q \in Q$, are homomorphisms of groups. First we notice that for $\alpha_1, \alpha_2 \in \mathrm{End}(G)$, we have

$$(\alpha_1 + \alpha_2)\Gamma_q^\star = \alpha_1\Gamma_q^\star + \alpha_2\Gamma_q^\star,$$

since for $s_i \in S$ where $s_i \in G_i$ for some $i \in \{0, 1, \ldots, n\}$, we have

$$\begin{aligned}
s_i(\alpha_1 + \alpha_2)\Gamma_q^\star &= s_i\underline{\alpha_1 + \alpha_2}_q \\
&= s_i\theta_i^{-1}(\alpha_1 + \alpha_2)\theta_{if_q} \\
&= s_i\theta_i^{-1}\alpha_1\theta_{if_q} + s_i\theta_i^{-1}\alpha_2\theta_{if_q} \\
&= s_i\underline{\alpha_1}_q + s_i\underline{\alpha_2}_q \\
&= s_i(\underline{\alpha_1}_q + \underline{\alpha_2}_q) \\
&= s_i(\alpha_1\Gamma_q^\star + \alpha_2\Gamma_q^\star).
\end{aligned}$$

Thus we only need to show that $(c)\Gamma_q^\star = 0$ whenever $0 = c \in E(G)$. Suppose that $c \in E(G)$ such that $c = \sum_{r=1}^d \epsilon_r\alpha_r = 0$, then $g\sum_{r=1}^d \epsilon_r\alpha_r = 0$ for all $g \in G$.

For $s_i \in S$ where $s_i \in G_i, i \in \{0, 1, \ldots, n\}$, there exists $g \in G$ such that $g = s_i\theta_i^{-1}$. Thus we have

$$\begin{aligned}
s_i\theta_i^{-1}\sum_{r=1}^d \epsilon_r\alpha_r &= 0 \\
s_i\sum_{r=1}^d \epsilon_r\theta_i^{-1}\alpha_r &= 0 \\
s_i(\sum_{r=1}^d \epsilon_r\theta_i^{-1}\alpha_r)\theta_{if_q} &= 0 \\
s_i(\sum_{r=1}^d \epsilon_r\theta_i^{-1}\alpha_r\theta_{if_q}) &= 0
\end{aligned}$$

$$
\begin{aligned}
s_i(\sum_{r=1}^{d} \epsilon_r \underline{\alpha_{r_q}}) &= 0 \\
s_i(\sum_{r=1}^{d} \epsilon_r \alpha_r)\Gamma_q^\star &= 0 \\
s_i(c)\Gamma_q^\star &= 0.
\end{aligned}
$$

Hence $(c)\Gamma_q^\star = 0$ and $\Gamma_q^\star, q \in Q$, are homomorphisms. Finally we show that $\Gamma_q^\star, q \in Q$, are monomorphisms. Let $0 \neq c = \sum_{r=1}^{d} \epsilon_r \alpha_r \in E(G)$. We show that $(c)\Gamma_q^\star \neq 0, q \in Q$. Since $c \neq 0$, there exists $g \in G$ such that $g \sum_{r=1}^{d} \epsilon_r \alpha_r \neq 0$. Consider the isomorphism $\theta_i : G \longrightarrow G_i, i \in \{0, 1, \ldots, n\}$. Then $(g)\theta_i = s_i$ for some $s_i \in G_i$. So we have

$$
\begin{aligned}
s_i(c)\Gamma_q^\star &= (s_i)(\sum_{r=1}^{d} \epsilon_r \alpha_r)\Gamma_q^\star \\
&= s_i(\sum_{r=1}^{d} \epsilon_r \underline{\alpha_{r_q}}) \\
&= g\theta_i(\sum_{r=1}^{d} \epsilon_r \underline{\alpha_{r_q}}) \\
&= g(\sum_{r=1}^{d} \epsilon_r \theta_i \underline{\alpha_{r_q}}) \\
&= g(\sum_{r=1}^{d} \epsilon_r \theta_i \theta_i^{-1} \alpha_r \theta_{if_q}) \\
&= g(\sum_{r=1}^{d} \epsilon_r \alpha_r \theta_{if_q}) \\
&= g(\sum_{r=1}^{d} \epsilon_r \alpha_r)\theta_{if_q} \\
&\neq 0 \text{ , since } \theta_{if_q} \text{ is an isomorphism.}
\end{aligned}
$$

Hence $(c)\Gamma_q^\star \neq 0$ showing that $\Gamma_q^\star, q \in Q$, are monomorphisms. We deduce that the endomorphisms of type f_q on S generate in $E(S)$ a subgroup isomorphic to the group $(E(G), +)$. It follows that there are L copies of $(E(G), +)$ lying inside $E(S)$ and briefly we can write

$$
(E(S), +) = \bigcup_{q \in Q} (E(G)^{f_q}, +) \tag{8.5}
$$

and $|\{E(G)^{f_q}\}| = L$.

The next step is to see how addition inside $E(S)$ behaves.

8.2 Addition in $E(S)$

In order to define addition between two different types of endomorphisms of S, we need to introduce the following endomorphisms on Y. For $f_t, f_w \in \mathrm{End}(Y)$, define a map

$$f_u : Y \longrightarrow Y$$

by

$$if_u \;=\; \min\{if_t, if_w\}. \tag{8.6}$$

The map f_u is an endomorphism of Y, for suppose that $j \leq i$ for $i, j \in Y$, then for the endomorphisms f_t, f_w, we have $jf_t \leq if_t$, and $jf_w \leq if_w$. So

$$jf_u = \min\{jf_t, jf_w\} \leq \min\{if_t, if_w\} = if_u.$$

Hence $jf_u \leq if_u$ and f_u is an endomorphism of Y.

The above endomorphism, f_u, could be considered as the sum (denoted by *) of endomorphisms f_t and f_w, and we write

$$f_t * f_w \;=\; f_u \tag{8.7}$$

Lemma 8.2.1 The endomorphisms of Y under the above sum form a semigroup.

Proof We need to show that the operation is associative. Let $f_t, f_w, f_v \in \mathrm{End}(Y)$, then

$$\begin{aligned}(f_t * f_w) * f_v \;&=\; \min\{f_t, f_w\} * f_v \\ &=\; \min\{\min\{f_t, f_w\}, f_v\} \\ &=\; \min\{f_t, f_w, f_v\}\end{aligned} \tag{8.8}$$

also we have

$$\begin{aligned}f_t * (f_w * f_v) \;&=\; f_t * \min\{f_w, f_v\} \\ &=\; \min\{f_t, \min\{f_w, f_v\}\} \\ &=\; \min\{f_t, f_w, f_v\}.\end{aligned} \tag{8.9}$$

Equations (8.8) and (8.9) imply that the operation * is associative and the endomorphisms of Y form a semigroup under this operation.

Remark 8.2.2 We observe that the operation * will induce an order, on the set End(Y), which is given by

$$f_{u_i} \geq f_{u_j} \iff f_{u_i} * f_{u_j} = f_{u_j}. \tag{8.10}$$

Now we consider the sum of two endomorphisms of different types on S. Let $\underline{\alpha}_t$ and $\underline{\beta}_w$ be two endomorphisms of type f_t and f_w respectively, where $t, w \in Q$. For $s_i \in S$ where $s_i \in G_i$ for some $i \in \{0, 1, \ldots, n\}$, we have

$$\begin{aligned}
s_i(\underline{\alpha}_t + \underline{\beta}_w) &= s_i\underline{\alpha}_t + s_i\underline{\beta}_w \\
&= s_i(\theta_i^{-1}\alpha\theta_{if_t}) + s_i(\theta_i^{-1}\beta\theta_{if_w}) \\
&= s_i(\theta_i^{-1}\alpha\theta_{if_t})\phi_{if_t,if_u} + s_i(\theta_i^{-1}\beta\theta_{if_w})\phi_{if_w,if_u} \text{ , where } f_u \text{ as defined in (8.6) ,} \\
&= s_i(\theta_i^{-1}\alpha)(\theta_{if_t})\phi_{if_t,if_u} + s_i(\theta_i^{-1}\beta)(\theta_{if_w})\phi_{if_w,if_u} \\
&= s_i(\theta_i^{-1}\alpha\theta_{if_u}) + s_i(\theta_i^{-1}\beta\theta_{if_u}) \\
&= s_i\underline{\alpha}_u + s_i\underline{\beta}_u \\
&= s_i(\underline{\alpha}_u + \underline{\beta}_u).
\end{aligned}$$

Thus

$$\underline{\alpha}_t + \underline{\beta}_w = \underline{\alpha}_u + \underline{\beta}_u \tag{8.11}$$

Similarly we can get

$$\underline{\beta}_w + \underline{\alpha}_t = \underline{\beta}_u + \underline{\alpha}_u \tag{8.12}$$

Equations (8.11) and (8.12) imply that the sum of $\underline{\alpha}_t$ with $\underline{\beta}_w$ lies in $E(G)^{f_u}$, where

$$f_u = f_t * f_w.$$

Now we use induction to deduce the following equation from equation (8.11).

$$\sum_{r=1}^{m} \epsilon_r \underline{\alpha_r}_t + \sum_{j=1}^{k} \eta_j \underline{\beta_j}_w = \sum_{r=1}^{m} \epsilon_r \underline{\alpha_r}_u + \sum_{j=1}^{k} \eta_j \underline{\beta_j}_u \tag{8.13}$$

where, $\epsilon_r = \pm 1$, $\eta_j = \pm 1$, $r = 1, \ldots, m$, $j = 1, \ldots, k$, and $\underline{\alpha_{r_t}}$, $\underline{\beta_{j_w}}$, $\underline{\alpha_{r_u}}$ and $\underline{\beta_{j_u}}$ are endomorphisms of S such that $f_u = f_t * f_w$. We can see that equation (8.13) is satisfied for $k = 1, m = 1$, since equation (8.11) gives

$$\epsilon_1 \underline{\alpha}_t + \eta_1 \underline{\beta}_w \;=\; \epsilon_1 \underline{\alpha}_u + \eta_1 \underline{\beta}_u. \tag{8.14}$$

In order to continue our induction process we should notice that if we replace $\underline{\beta}_w$ by $\underline{\beta}_u$ in equation (8.14), then easily we can get

$$\epsilon_1 \underline{\alpha}_t + \eta_1 \underline{\beta}_u \;=\; \epsilon_1 \underline{\alpha}_u + \eta_1 \underline{\beta}_u \tag{8.15}$$

which will imply that

$$\sum_{r=1}^{m} \epsilon_r \underline{\alpha_{r_t}} + \eta_1 \underline{\beta_{1_u}} \;=\; \sum_{r=1}^{m} \epsilon_r \underline{\alpha_{r_u}} + \eta_1 \underline{\beta_{1_u}}. \tag{8.16}$$

Also if we replace $\underline{\alpha}_t$ by $\underline{\beta}_u$ in equation (8.14), then we will get

$$\epsilon_1 \underline{\beta}_u + \eta_1 \underline{\beta}_w \;=\; \epsilon_1 \underline{\beta}_u + \eta_1 \underline{\beta}_u. \tag{8.17}$$

Furthermore,

$$\begin{aligned}
\sum_{r=1}^{m} \epsilon_r \underline{\alpha_{r_t}} + \eta_1 \underline{\beta_{1_w}} &= \sum_{r=1}^{m-1} \epsilon_r \underline{\alpha_{r_t}} + \epsilon_m \underline{\alpha_{m_t}} + \eta_1 \underline{\beta_{1_w}} \\
&= \sum_{r=1}^{m-1} \epsilon_r \underline{\alpha_{r_t}} + \epsilon_m \underline{\alpha_{m_u}} + \eta_1 \underline{\beta_{1_u}}, \text{by (8.14)}, \\
&= \sum_{r=1}^{m-1} \epsilon_r \underline{\alpha_{r_u}} + \epsilon_m \underline{\alpha_{m_u}} + \eta_1 \underline{\beta_{1_u}}, \text{by (8.16)}, \\
&= \sum_{r=1}^{m} \epsilon_r \underline{\alpha_{r_u}} + \eta_1 \underline{\beta_{1_u}}.
\end{aligned} \tag{8.18}$$

Now we apply induction on k in equation (8.13). First it is clear from equation (8.18) that equation (8.13) is valid for $k = 1$. Suppose that equation (8.13) is satisfied for $k - 1$, then

$$\begin{aligned}
\sum_{r=1}^{m} \epsilon_r \underline{\alpha_{r_t}} + \sum_{j=1}^{k} \eta_j \underline{\beta_{j_w}} &= \sum_{r=1}^{m} \epsilon_r \underline{\alpha_{r_t}} + \sum_{j=1}^{k-1} \eta_j \underline{\beta_{j_w}} + \eta_k \underline{\beta_{k_w}} \\
&= \sum_{r=1}^{m} \epsilon_r \underline{\alpha_{r_u}} + \sum_{j=1}^{k-1} \eta_j \underline{\beta_{j_u}} + \eta_k \underline{\beta_{k_w}}, \text{ by assumption},
\end{aligned}$$

$$= \sum_{r=1}^{m} \epsilon_r \underline{\alpha_r}_u + \sum_{j=1}^{k-2} \eta_j \underline{\beta_j}_u + \eta_{k-1} \underline{\beta_{k-1}}_u + \eta_k \underline{\beta_k}_w$$

$$= \sum_{r=1}^{m} \epsilon_r \underline{\alpha_r}_u + \sum_{j=1}^{k-2} \eta_j \underline{\beta_j}_u + \eta_{k-1} \underline{\beta_{k-1}}_u + \eta_k \underline{\beta_k}_u$$

$$= \sum_{r=1}^{m} \epsilon_r \underline{\alpha_r}_u + \sum_{j=1}^{k} \eta_j \underline{\beta_j}_u .$$

Hence equation (8.13) is satisfied. Similarly we can show that

$$\sum_{j=1}^{k} \eta_j \underline{\beta_j}_w + \sum_{r=1}^{m} \epsilon_r \underline{\alpha_r}_t \;=\; \sum_{j=1}^{k} \eta_j \underline{\beta_j}_u + \sum_{r=1}^{m} \epsilon_r \underline{\alpha_r}_u . \tag{8.19}$$

Equations (8.13) and (8.19) show that the sum of an element in $E(G)^{f_t}$ with an element in $E(G)^{f_w}$ lies in $E(G)^{f_u}$ where $f_u = f_t * f_w$. In order to get a precise description of the general sum in $E(S)$, we give the following definition which is a general form of (8.7) :

Definition 8.2.3 For $2 \leq k \leq m$,

$$f_{u(k)} \;=\; *_{i=1}^{k} f_{q_i} . \tag{8.20}$$

Now we will write equations (8.13) and (8.19) in a more general way. In fact, we can obtain the following :

$$\sum_{r=1}^{n_1} \epsilon_{1r} \underline{\alpha_{1r}}_{q_1} + \sum_{r=1}^{n_2} \epsilon_{2r} \underline{\alpha_{2r}}_{q_2} + \ldots + \sum_{r=1}^{n_m} \epsilon_{mr} \underline{\alpha_{mr}}_{q_m} = \tag{8.21}$$

$$= \sum_{r=1}^{n_1} \epsilon_{1r} \underline{\alpha_{1r}}_{q_v} + \ldots + \sum_{r=1}^{n_m} \epsilon_{mr} \underline{\alpha_{mr}}_{q_v}$$

where $\epsilon_{kr} = \pm 1$, $k = 1, \ldots, m$, $r = 1, \ldots, n_k$, and

$$f_{q_v} \;=\; f_{u(m)} \tag{8.22}$$

Using the definition of addition which is applied in (8.13) and (8.19), with the above notation, we can see that equation (8.21) is satisfied as follows :

$\sum_{r=1}^{n_1} \epsilon_{1r} \underline{\alpha_{1r}}_{q_1} + \sum_{r=1}^{n_2} \epsilon_{2r} \underline{\alpha_{2r}}_{q_2} + \sum_{r=1}^{n_3} \epsilon_{3r} \underline{\alpha_{3r}}_{q_3} + \ldots + \sum_{r=1}^{n_m} \epsilon_{mr} \underline{\alpha_{mr}}_{q_m} =$

$= \sum_{r=1}^{n_1} \epsilon_{1r} \underline{\alpha_{1r}}_{u(2)} + \sum_{r=1}^{n_2} \epsilon_{2r} \underline{\alpha_{2r}}_{u(2)} + \sum_{r=1}^{n_3} \epsilon_{3r} \underline{\alpha_{3r}}_{q_3} + \ldots + \sum_{r=1}^{n_m} \epsilon_{mr} \underline{\alpha_{mr}}_{q_m}$

$= \sum_{r=1}^{n_1} \epsilon_{1r} \underline{\alpha_{1r}}_{u(3)} + \sum_{r=1}^{n_2} \epsilon_{2r} \underline{\alpha_{2r}}_{u(3)} + \sum_{r=1}^{n_3} \epsilon_{3r} \underline{\alpha_{3r}}_{u(3)} + \ldots + \sum_{r=1}^{n_m} \epsilon_{mr} \underline{\alpha_{mr}}_{q_m}$

$\vdots$

$= \sum_{r=1}^{n_1} \epsilon_{1r} \underline{\alpha_{1r}}_{u(m-1)} + \sum_{r=1}^{n_2} \epsilon_{2r} \underline{\alpha_{2r}}_{u(m-1)} + \sum_{r=1}^{n_3} \epsilon_{3r} \underline{\alpha_{3r}}_{u(m-1)} + \ldots +$
$+ \sum_{r=1}^{n_{(m-1)}} \epsilon_{(m-1)r} \underline{\alpha_{(m-1)r}}_{u(m-1)} + \sum_{r=1}^{n_m} \epsilon_{mr} \underline{\alpha_{mr}}_{q_m}$
$= \sum_{r=1}^{n_1} \epsilon_{1r} \underline{\alpha_{1r}}_{u(m)} + \sum_{r=1}^{n_2} \epsilon_{2r} \underline{\alpha_{2r}}_{u(m)} + \ldots + \sum_{r=1}^{n_{(m-1)}} \epsilon_{(m-1)r} \underline{\alpha_{(m-1)r}}_{u(m)} + \sum_{r=1}^{n_m} \epsilon_{mr} \underline{\alpha_{mr}}_{u(m)}$
$= \sum_{r=1}^{n_1} \epsilon_{1r} \underline{\alpha_{1r}}_{q_v} + \sum_{r=1}^{n_2} \epsilon_{2r} \underline{\alpha_{2r}}_{q_v} + \ldots + \sum_{r=1}^{n_m} \epsilon_{mr} \underline{\alpha_{mr}}_{q_v}$
where, $q_v = u(m)$, and f_{q_v} is determined as in (8.22).

Hence equation (8.21) is satisfied, which shows that the sum of different elements in $E(S)$ will lie in $E(G)^{f_{q_v}}$, the near-ring generated by the endomorphisms of type f_{q_v}, where $f_{q_v} = f_{u(m)} = *_{i=1}^{m} f_{q_i}$.

The next step now is to look at the product inside $E(S)$.

8.3 Product in $E(S)$

Let $\underline{\alpha}_t$ and $\underline{\beta}_w$ be two endomorphisms of type f_t and f_w respectively, where $t, w \in Q$. For $s_i \in S$ where $s_i \in G_i$ for some $i \in \{0, 1, \ldots, n\}$, we have

$$\begin{aligned} s_i(\underline{\alpha}_t \underline{\beta}_w) &= s_i(\theta_i^{-1} \alpha \theta_{if_t}) \underline{\beta}_w \\ &= s_i(\theta_i^{-1} \alpha \theta_{if_t})(\theta_{if_t}^{-1} \beta \theta_{if_t f_w}) \\ &= s_i(\theta_i^{-1} \alpha\beta \theta_{if_t f_w}) \\ &= s_i(\theta_i^{-1} \alpha\beta \theta_{if_{t\bullet w}}) \text{ , where } f_{t\bullet w} \text{ is the product } f_t f_w \text{ in End}(Y), \\ &= s_i(\underline{\alpha\beta}_{t\bullet w}). \end{aligned}$$

Thus

$$\underline{\alpha}_t \underline{\beta}_w = \underline{\alpha\beta}_{t\bullet w}. \tag{8.23}$$

If we reverse the order of the maps in the above product, we get

$$\begin{aligned} s_i(\underline{\beta}_w \underline{\alpha}_t) &= s_i(\theta_i^{-1} \beta \theta_{if_w}) \underline{\alpha}_t \\ &= s_i(\theta_i^{-1} \beta \theta_{if_w})(\theta_{if_w}^{-1} \alpha \theta_{if_w f_t}) \\ &= s_i(\theta_i^{-1} \beta\alpha \theta_{if_{w\bullet t}}) \\ &= s_i(\underline{\beta\alpha}_{w\bullet t}). \end{aligned}$$

So

$$\underline{\beta}_w \underline{\alpha}_t \;=\; \underline{\beta\alpha}_{w\bullet t}. \tag{8.24}$$

Equations (8.23) and (8.24) imply that the product $\underline{\alpha}_t \underline{\beta}_w$ does not lie in the same near-ring as the product $\underline{\beta}_w \underline{\alpha}_t$ lies unless $f_{t\bullet w} = f_{w\bullet t}$, which is not true in general. It follows that the product here fails to give a semilattice of semigroups.

We now return to the sum in $E(S)$ to get our result.

8.4 Conclusion IV

Recall the endomorphism f_u defined in (8.6) with remark (8.2.2), then an endomorphism of Y could be an endomorphism of the form f_u and we can write

$$\text{End}(Y) = \{f_{u_1}, f_{u_2}, \ldots, f_{u_L}\} = \mathcal{L}, \text{ say.}$$

Consider now the semilattice $\mathcal{L}$ and define the homomorphisms $\phi_{f_{u_i},f_{u_j}}$ such that for $f_{u_i} \geq f_{u_j}$;

$$\phi_{f_{u_i},f_{u_j}} : E(G)^{f_{u_i}} \longrightarrow E(G)^{f_{u_j}}$$

is defined by

$$\textstyle\sum_r \epsilon_r \underline{\alpha_r}_{u_i} \longrightarrow \sum_r \epsilon_r \underline{\alpha_r}_{u_j}.$$

Then we have a strong semilattice of groups given by

$$E(S) = (\mathcal{L} \,, \{E(G)^{f_q}\}_{f_q \in \mathcal{L}} \,,\, \{\phi_{f_{u_i},f_{u_j}}\})$$

that is,

$(E(S), +)$ is a Clifford semigroup.

Chapter 9

Seminear-ring of endomorphisms V

In this chapter we consider the case in which not all the elements of the semilattice are comparable.

9.1 Starting case V

Let

$$Y = \{0, 1, 2\}$$

be a semilattice with $1 \geq 0$ and $2 \geq 0$ (1 and 2 are not comparable).

Let G_0, G_1 and G_2 be isomorphic groups, i.e.

$$G_0 \cong G_1 \cong G_2$$

Let $\phi_{1,0}$ and $\phi_{2,0}$ be the isomorphisms such that

$$\phi_{1,0} : G_1 \longrightarrow G_0$$

and

$$\phi_{2,0} : G_2 \longrightarrow G_0.$$

Let G be a group that is isomorphic to the groups G_0, G_1 and G_2. So we can consider the isomorphisms

$$\theta_i : G \longrightarrow G_i\,, i = 0, 1, 2.$$

The isomorphisms θ_1 and θ_2 can be defined by

$$\theta_0 = \theta_k\phi_{k,0} \tag{9.1}$$

where $k = 1, 2$.

Consider the above semilattice Y, then the endomorphisms of Y will consist of all elements of the set

$$\mathrm{End}(Y) = \{f_q\,;\,q \in \{1, 2, \ldots, 11\}\}$$

where these endomorphisms are given on the next page.

If $f \in \mathrm{End}(S)$ satisfies $f|_Y = f_q$ then we call f an endomorphism of type f_q. Consider $S = \bigcup_{i=0,1,2} G_i$. By theorem 4.2.3, for an endomorphism f of S, $f|_Y$ is an endomorphism of Y, and for $i, j \in \{0, 1, 2\}$, we have

$$f|_{G_i} \in \mathrm{End}(G_i) \cong \mathrm{End}(G) \quad \text{or} \quad f|_{G_i} \in \mathrm{Hom}(G_i, G_j) \cong \mathrm{End}(G),$$

where $j = if$. It should be noted that the first one of the above isomorphisms is an isomorphism of semigroups while the second one is a 1-1 correspondence. Therefore, for each endomorphism of S of type $f_q, q \in \{1, 2, \ldots, 11\}$, there will be an endomorphism of G, and the following correspondence holds :

$$\{f|_{G_2}\,;\,f \in \mathrm{End}(S)\} \longleftrightarrow \{f|_{G_1}\,;\,f \in \mathrm{End}(S)\} \longleftrightarrow \{f|_{G_0}\,;\,f \in \mathrm{End}(S)\} \longleftrightarrow \mathrm{End}(G).$$

Let α be an endomorphism of G. With an endomorphism $f_q \in \mathrm{End}(Y)$, define a map

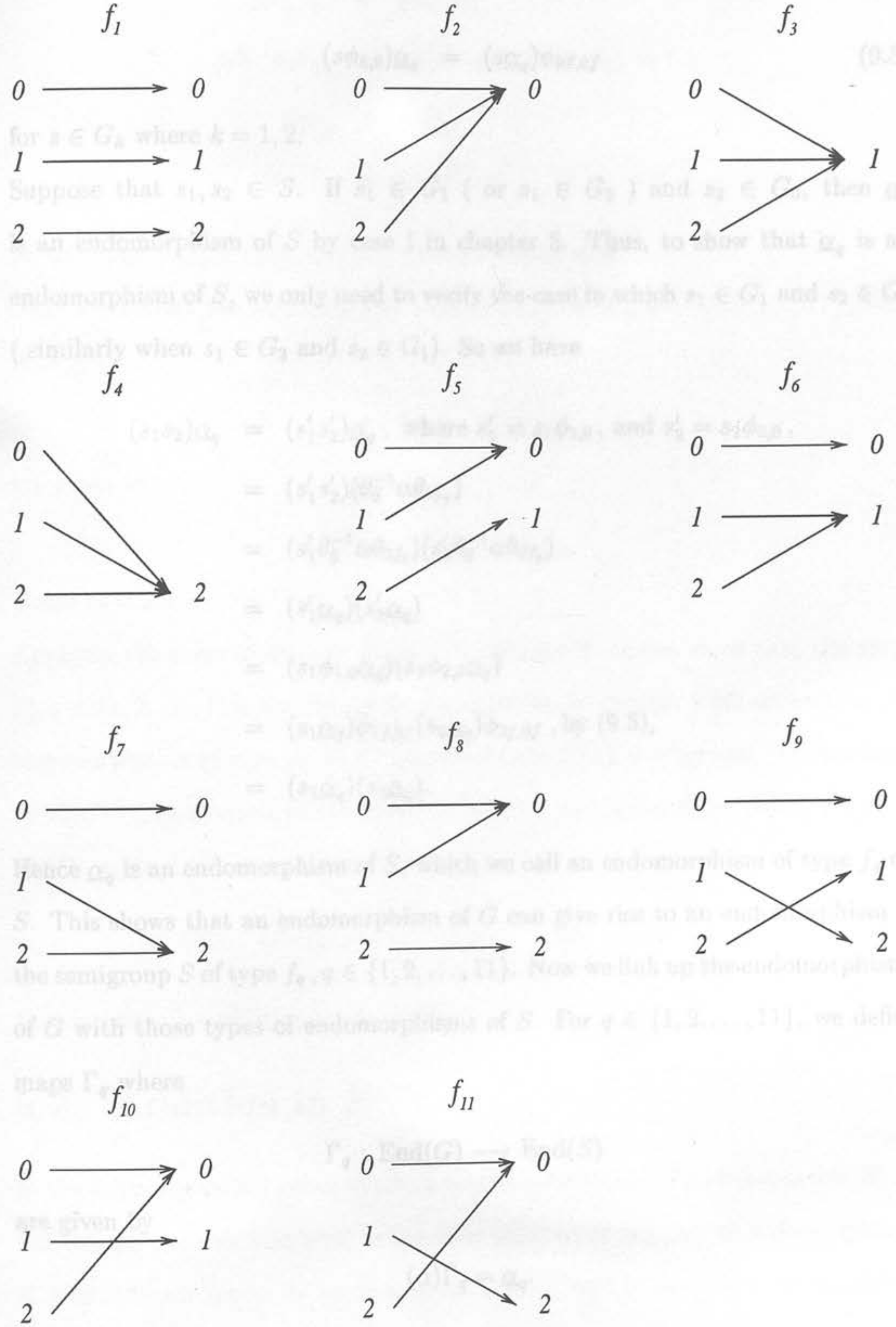
$$\underline{\alpha}_q : S \longrightarrow S$$

by

$$(s_i)\underline{\alpha}_q = s_i\theta_i^{-1}\alpha\theta_{if_q} \tag{9.2}$$

for $s \in G_i$, where $i = 0, 1, 2$.

We will show that the map $\underline{\alpha}_q$ is an endomorphism of S. First we notice that a



f1
f2
f3
f4
f5
f6
f7
f8
f9
f10
f11
0
1
2

similar argument to that in proposition 8.1.3 can be used to obtain the following equation

$$(s\phi_{k,0})\underline{\alpha}_q = (s\underline{\alpha}_q)\phi_{kf,0f} \tag{9.3}$$

for $s \in G_k$ where $k = 1, 2$.

Suppose that $s_1, s_2 \in S$. If $s_1 \in G_1$ (or $s_1 \in G_2$) and $s_2 \in G_0$, then $\underline{\alpha}_q$ is an endomorphism of S by case I in chapter 5. Thus, to show that $\underline{\alpha}_q$ is an endomorphism of S, we only need to verify the case in which $s_1 \in G_1$ and $s_2 \in G_2$ (similarly when $s_1 \in G_2$ and $s_2 \in G_1$). So we have

$$\begin{aligned}
(s_1s_2)\underline{\alpha}_q &= (s_1's_2')\underline{\alpha}_q \text{ , where } s_1' = s_1\phi_{1,0} \text{ , and } s_2' = s_2\phi_{2,0} \text{ ,} \\
&= (s_1's_2')(\theta_0^{-1}\alpha\theta_{0f_q}) \\
&= (s_1'\theta_0^{-1}\alpha\theta_{0f_q})(s_2'\theta_0^{-1}\alpha\theta_{0f_q}) \\
&= (s_1'\underline{\alpha}_q)(s_2'\underline{\alpha}_q) \\
&= (s_1\phi_{1,0}\underline{\alpha}_q)(s_2\phi_{2,0}\underline{\alpha}_q) \\
&= (s_1\underline{\alpha}_q)\phi_{1f,0f}(s_2\underline{\alpha}_q)\phi_{2f,0f} \text{ , by (9.3),} \\
&= (s_1\underline{\alpha}_q)(s_2\underline{\alpha}_q).
\end{aligned}$$

Hence $\underline{\alpha}_q$ is an endomorphism of S, which we call an endomorphism of type f_q on S. This shows that an endomorphism of G can give rise to an endomorphism of the semigroup S of type f_q, $q \in \{1, 2, \ldots, 11\}$. Now we link up the endomorphisms of G with those types of endomorphisms of S. For $q \in \{1, 2, \ldots, 11\}$, we define maps Γ_q where

$$\Gamma_q : \mathrm{End}(G) \longrightarrow \mathrm{End}(S)$$

are given by

$$(\alpha)\Gamma_q = \underline{\alpha}_q.$$

We show that the maps $\Gamma_q, q \in Q$, are one-to-one. Suppose that $\alpha_1, \alpha_2 \in \mathrm{End}(G)$ such that $\alpha_1 \neq \alpha_2$, then there exists $g \in G$ such that $g\alpha_1 \neq g\alpha_2$. But $g = g_i\theta_i^{-1}$

for some $g_i \in G_i$, where $i \in \{0, 1, 2\}$. Thus we have

$$\begin{aligned}
g_i\theta_i^{-1}\alpha_1 &\neq g_i\theta_i^{-1}\alpha_2, \\
g_i\theta_i^{-1}\alpha_1\theta_{if_q} &\neq g_i\theta_i^{-1}\alpha_2\theta_{if_q}, \text{since } \theta_{ifq} \text{ is 1-1}, \\
g_i\underline{\alpha_1}_q &\neq g_i\underline{\alpha_2}_q, \\
(\alpha_1)\Gamma_q &\neq (\alpha_2)\Gamma_q.
\end{aligned}$$

So the maps $\Gamma_q, q \in \{1, 2, \ldots, 11\}$, are one-to-one maps.

Next we extend the maps $\Gamma_q, q \in \{1, 2, \ldots, 11\}$, to $\Gamma_q^\star, q \in \{1, 2, \ldots, 11\}$, such that

$$\Gamma_q^\star : E(G) \longrightarrow E(S)$$

are given by

$$(\sum \epsilon_r\alpha_r)\Gamma_q^\star = \sum \epsilon_r\underline{\alpha_r}_q$$

where $\epsilon_r = \pm 1$.

Applying the same method as in case IV, chapter 8, we can show that the maps $\Gamma_q^\star, q \in \{1, 2, \ldots, 11\}$, are indeed monomorphisms of groups. This shows that the endomorphisms of type f_q on S will generate in $E(S)$ a subgroup, (we denote it by $E(G)^{f_q}$), isomorphic to the group $(E(G), +)$, and hence there are 11 copies of $(E(G), +)$ in $E(S)$ and we can write

$$(E(S), +) = \bigcup_{q \in \{1,2,\ldots,11\}} (E(G)^{f_q}, +). \tag{9.4}$$

Now we describe addition inside $E(S)$.

9.2 Addition in $E(S)$

In the following we are going to give a precise description of addition inside $E(S)$. For this purpose we first need to consider the sum of any pair of endomorphisms of S of different types. So let us suppose that $\underline{\alpha}_3$ and $\underline{\beta}_7$ are two endomorphisms of S of type f_3 and type f_7 respectively. Then for $g_2 \in G_2$, we have

$$g_2(\underline{\alpha}_3 + \underline{\beta}_7) = g_2\underline{\alpha}_3 + g_2\underline{\beta}_7$$

$$
\begin{aligned}
&= g_2(\theta_2^{-1}\alpha\theta_{2f_3}) + g_2(\theta_2^{-1}\beta\theta_{2f_7}) \\
&= g_2(\theta_2^{-1}\alpha\theta_1) + g_2(\theta_2^{-1}\beta\theta_2) \\
&= g_2(\theta_2^{-1}\alpha\theta_1)\phi_{1,0} + g_2(\theta_2^{-1}\beta\theta_2)\phi_{2,0} \\
&= g_2(\theta_2^{-1}\alpha\theta_0) + g_2(\theta_2^{-1}\beta\theta_0) \\
&= g_2(\theta_2^{-1}\alpha\theta_{2f_2}) + g_2(\theta_2^{-1}\beta\theta_{2f_2}) \\
&= g_2\underline{\alpha}_2 + g_2\underline{\beta}_2 \\
&= g_2(\underline{\alpha}_2 + \underline{\beta}_2). \qquad (9.5)
\end{aligned}
$$

Similarly we can get

$$g_2(\underline{\beta}_7 + \underline{\alpha}_3) = g_2(\underline{\beta}_2 + \underline{\alpha}_2). \qquad (9.6)$$

Now for $g_1 \in G_1$, we have

$$
\begin{aligned}
g_1(\underline{\alpha}_3 + \underline{\beta}_7) &= g_1\underline{\alpha}_3 + g_1\underline{\beta}_7 \\
&= g_1(\theta_1^{-1}\alpha\theta_{1f_3}) + g_1(\theta_1^{-1}\beta\theta_{1f_7}) \\
&= g_1(\theta_1^{-1}\alpha\theta_1) + g_1(\theta_1^{-1}\beta\theta_2) \\
&= g_1(\theta_1^{-1}\alpha\theta_1)\phi_{1,0} + g_1(\theta_1^{-1}\beta\theta_2)\phi_{2,0} \\
&= g_1(\theta_1^{-1}\alpha\theta_0) + g_1(\theta_1^{-1}\beta\theta_0) \\
&= g_1(\theta_1^{-1}\alpha\theta_{1f_2}) + g_1(\theta_1^{-1}\beta\theta_{1f_2}) \\
&= g_1\underline{\alpha}_2 + g_1\underline{\beta}_2 \\
&= g_1(\underline{\alpha}_2 + \underline{\beta}_2). \qquad (9.7)
\end{aligned}
$$

Similarly we can get

$$g_1(\underline{\beta}_7 + \underline{\alpha}_3) = g_1(\underline{\beta}_2 + \underline{\alpha}_2). \qquad (9.8)$$

Finally for $g_0 \in G_0$, we have

$$
\begin{aligned}
g_0(\underline{\alpha}_3 + \underline{\beta}_7) &= g_0\underline{\alpha}_3 + g_0\underline{\beta}_7 \\
&= g_0(\theta_0^{-1}\alpha\theta_{0f_3}) + g_0(\theta_0^{-1}\beta\theta_{0f_7}) \\
&= g_0(\theta_0^{-1}\alpha\theta_1) + g_0(\theta_0^{-1}\beta\theta_0)
\end{aligned}
$$

$$
\begin{aligned}
&= g_0(\theta_0^{-1}\alpha\theta_1)\phi_{1,0} + g_0(\theta_0^{-1}\beta\theta_0) \\
&= g_0(\theta_0^{-1}\alpha\theta_0) + g_0(\theta_0^{-1}\beta\theta_0) \\
&= g_0(\theta_0^{-1}\alpha\theta_{0f_2}) + g_0(\theta_0^{-1}\beta\theta_{0f_2}) \\
&= g_0\underline{\alpha}_2 + g_0\underline{\beta}_2 \\
&= g_0(\underline{\alpha}_2 + \underline{\beta}_2).
\end{aligned} \tag{9.9}
$$

Similarly we can get

$$
g_0(\underline{\beta}_7 + \underline{\alpha}_3) \;=\; g_0(\underline{\beta}_2 + \underline{\alpha}_2). \tag{9.10}
$$

Equations (9.5) — (9.10) show that the sum of two endomorphisms of type f_3 and type f_7 lies in $E(G)^{f_2}$. Following the same pattern, we can obtain the following table of addition :

$+$	f_1	f_2	f_3	f_4	f_5	f_6	f_7	f_8	f_9	f_{10}	f_{11}
f_1	1	2	10	8	2	10	8	8	2	10	2
f_2	2	2	2	2	2	2	2	2	2	2	2
f_3	10	2	3	2	5	6	2	2	5	10	2
f_4	8	2	2	4	2	2	7	8	11	2	11
f_5	2	2	5	2	5	5	2	2	5	2	2
f_6	10	2	6	2	5	6	2	2	5	10	2
f_7	8	2	2	7	2	2	7	8	11	2	11
f_8	8	2	2	8	2	2	8	8	2	2	2
f_9	2	2	5	11	5	5	11	2	9	2	11
f_{10}	10	2	10	2	2	10	2	2	2	10	2
f_{11}	2	2	2	11	2	2	11	2	11	2	11

The number q in the above table, and in the subsequent tables, denotes the endomorphism f_q.

We can generalize the sum in the above table as follows. Consider, for example, the sum $\underline{\alpha}_1 + \underline{\beta}_2$, then from the above table we know that

$$
\underline{\alpha}_1 + \underline{\beta}_2 \;=\; \underline{\alpha}_2 + \underline{\beta}_2. \tag{9.11}
$$

Using induction, from the above equation, we can obtain the following

$$
\sum_{r=1}^{m} \epsilon_r \underline{\alpha_{r}}_1 + \sum_{j=1}^{k} \eta_j \underline{\beta_{j}}_2 \;=\; \sum_{r=1}^{m} \epsilon_r \underline{\alpha_{r}}_2 + \sum_{j=1}^{k} \eta_j \underline{\beta_{j}}_2 \tag{9.12}
$$

where, $\underline{\alpha_{r_1}}$ are endomorphisms of type f_1 and $\underline{\beta_{j_2}}$ are endomorphisms of type f_2 and $\epsilon_r = \pm 1$, $\eta_j = \pm 1$, $r = 1, \ldots, m$, $j = 1, \ldots, k$.

Clearly equation (9.12) is true for $m = 1, k = 1$ as it is seen in (9.11). A simple induction on m, can show that

$$\sum_{r=1}^{m} \epsilon_r \underline{\alpha_{r_1}} + \eta_1 \underline{\beta_2} = \sum_{r=1}^{m} \epsilon_r \underline{\alpha_{r_2}} + \eta_1 \underline{\beta_2} \tag{9.13}$$

So we only need to apply induction on k in equation (9.12). Suppose that equation (9.12) is true for $k-1$, then we have

$$\begin{aligned}
\sum_{r=1}^{m} \epsilon_r \underline{\alpha_{r_1}} + \sum_{j=1}^{k} \eta_j \underline{\beta_{j_2}} &= \sum_{r=1}^{m} \epsilon_r \underline{\alpha_{r_1}} + \sum_{j=1}^{k-1} \eta_j \underline{\beta_{j_2}} + \eta_k \underline{\beta_{k_2}} \\
&= \sum_{r=1}^{m} \epsilon_r \underline{\alpha_{r_2}} + \sum_{j=1}^{k-1} \eta_j \underline{\beta_{j_2}} + \eta_k \underline{\beta_{k_2}}, \text{by hypothesis,} \\
&= \sum_{r=1}^{m} \epsilon_r \underline{\alpha_{r_2}} + \sum_{j=1}^{k} \eta_j \underline{\beta_{j_2}}.
\end{aligned}$$

Hence equation (9.12) is satisfied. Since the sum is not affected when reversing the order of the elements, we can also get

$$\sum_{j=1}^{k} \eta_j \underline{\beta_{j_2}} + \sum_{r=1}^{m} \epsilon_r \underline{\alpha_{r_1}} = \sum_{j=1}^{k} \eta_j \underline{\beta_{j_2}} + \sum_{r=1}^{m} \epsilon_r \underline{\alpha_{r_2}}. \tag{9.14}$$

This shows that the sum of an element in $E(G)^{f_1}$ with an element in $E(G)^{f_2}$ lies in $E(G)^{f_2}$. Similar arguments can be applied to obtain all the sums in $E(S)$ which is described in the following table :

$+$	$E(G)^1$	$E(G)^2$	$E(G)^3$	$E(G)^4$	$E(G)^5$	$E(G)^6$	$E(G)^7$	$E(G)^8$	$E(G)^9$	$E(G)^{10}$	$E(G)^{11}$
$E(G)^1$	$E(G)^1$	$E(G)^2$	$E(G)^{10}$	$E(G)^8$	$E(G)^2$	$E(G)^{10}$	$E(G)^8$	$E(G)^8$	$E(G)^2$	$E(G)^{10}$	$E(G)^2$
$E(G)^2$	$E(G)^2$	$E(G)^2$	$E(G)^2$	$E(G)^2$	$E(G)^2$	$E(G)^2$	$E(G)^2$	$E(G)^2$	$E(G)^2$	$E(G)^2$	$E(G)^2$
$E(G)^3$	$E(G)^{10}$	$E(G)^2$	$E(G)^3$	$E(G)^2$	$E(G)^5$	$E(G)^6$	$E(G)^2$	$E(G)^2$	$E(G)^5$	$E(G)^{10}$	$E(G)^2$
$E(G)^4$	$E(G)^8$	$E(G)^2$	$E(G)^2$	$E(G)^4$	$E(G)^2$	$E(G)^2$	$E(G)^7$	$E(G)^8$	$E(G)^{11}$	$E(G)^2$	$E(G)^{11}$
$E(G)^5$	$E(G)^2$	$E(G)^2$	$E(G)^5$	$E(G)^2$	$E(G)^5$	$E(G)^5$	$E(G)^2$	$E(G)^2$	$E(G)^5$	$E(G)^2$	$E(G)^2$
$E(G)^6$	$E(G)^{10}$	$E(G)^2$	$E(G)^6$	$E(G)^2$	$E(G)^5$	$E(G)^6$	$E(G)^2$	$E(G)^2$	$E(G)^5$	$E(G)^{10}$	$E(G)^2$
$E(G)^7$	$E(G)^8$	$E(G)^2$	$E(G)^2$	$E(G)^7$	$E(G)^2$	$E(G)^2$	$E(G)^7$	$E(G)^8$	$E(G)^{11}$	$E(G)^2$	$E(G)^{11}$
$E(G)^8$	$E(G)^8$	$E(G)^2$	$E(G)^2$	$E(G)^8$	$E(G)^2$	$E(G)^2$	$E(G)^8$	$E(G)^8$	$E(G)^2$	$E(G)^2$	$E(G)^2$
$E(G)^9$	$E(G)^2$	$E(G)^2$	$E(G)^5$	$E(G)^{11}$	$E(G)^5$	$E(G)^5$	$E(G)^{11}$	$E(G)^2$	$E(G)^9$	$E(G)^2$	$E(G)^{11}$
$E(G)^{10}$	$E(G)^{10}$	$E(G)^2$	$E(G)^{10}$	$E(G)^2$	$E(G)^2$	$E(G)^{10}$	$E(G)^2$	$E(G)^2$	$E(G)^2$	$E(G)^{10}$	$E(G)^2$
$E(G)^{11}$	$E(G)^2$	$E(G)^2$	$E(G)^2$	$E(G)^{11}$	$E(G)^2$	$E(G)^2$	$E(G)^{11}$	$E(G)^2$	$E(G)^{11}$	$E(G)^2$	$E(G)^{11}$

Next, as might be expected, we look at the product inside $E(S)$.

9.3 Product in $E(S)$

Let us start by considering $\underline{\alpha}_8$ and $\underline{\beta}_{10}$ as two endomorphisms of S of type f_8 and type f_{10} respectively. Then for $g_2 \in G_2$, we have

$$\begin{aligned}
g_2(\underline{\alpha}_8\underline{\beta}_{10}) &= g_2(\theta_2^{-1}\alpha\theta_{2f_8})\underline{\beta}_{10} \\
&= g_2(\theta_2^{-1}\alpha\theta_2)\underline{\beta}_{10} \\
&= g_2(\theta_2^{-1}\alpha\theta_2)(\theta_2^{-1}\beta\theta_{2f_{10}}) \\
&= g_2(\theta_2^{-1}\alpha\beta\theta_{2f_{10}}) \\
&= g_2(\theta_2^{-1}\alpha\beta\theta_0) \\
&= g_2(\theta_2^{-1}\alpha\beta\theta_{2f_2}) \\
&= g_2\underline{\alpha\beta}_2. \qquad (9.15)
\end{aligned}$$

Also we have

$$\begin{aligned}
g_2(\underline{\beta}_{10}\underline{\alpha}_8) &= g_2(\theta_2^{-1}\beta\theta_{2f_{10}})\underline{\alpha}_8 \\
&= g_2(\theta_2^{-1}\beta\theta_0)\underline{\alpha}_8 \\
&= g_2(\theta_2^{-1}\beta\theta_0)(\theta_0^{-1}\alpha\theta_{0f_8}) \\
&= g_2(\theta_2^{-1}\beta\alpha\theta_{0f_8}) \\
&= g_2(\theta_2^{-1}\beta\alpha\theta_0) \\
&= g_2(\theta_2^{-1}\beta\alpha\theta_{2f_2}) \\
&= g_2\underline{\alpha\beta}_2. \qquad (9.16)
\end{aligned}$$

Equations (9.15) and (9.16) can also be obtained if we replace g_2 by g_1 or g_0. Hence the product of $\underline{\alpha}_8$ with $\underline{\beta}_{10}$ lies in $E(G)^{f_2}$. The same pattern can be used to obtain the following table of products.

$\cdot$	f_1	f_2	f_3	f_4	f_5	f_6	f_7	f_8	f_9	f_{10}	f_{11}
f_1	1	2	3	4	5	6	7	8	9	10	11
f_2	2	2	3	4	2	2	2	2	2	2	2
f_3	3	2	3	4	2	3	4	2	4	3	4
f_4	4	2	3	4	3	3	4	4	3	2	2
f_5	5	2	3	4	2	5	8	2	8	5	8
f_6	6	2	3	4	2	6	7	2	7	6	7
f_7	7	2	3	4	6	6	7	7	6	2	2
f_8	8	2	3	4	5	5	8	8	5	2	2
f_9	9	2	3	4	10	6	7	11	1	5	8
f_{10}	10	2	3	4	2	10	11	2	11	10	11
f_{11}	11	2	3	4	10	10	11	11	10	2	2

We can see from the product table that if $\underline{\delta}_4$ and $\underline{\omega}_5$ are two endomorphisms of type f_4 and f_5 respectively, then

$$\underline{\delta_4\omega_5} = \underline{\delta\omega_3} \quad \text{while} \quad \underline{\omega_5\delta_4} = \underline{\omega\delta_4}.$$

This comes from the fact that $f_4 f_5 \neq f_5 f_4$. It follows that the product $\underline{\delta_4\omega_5}$ lies in $E(G)^{f_3}$ while the product $\underline{\omega_5\delta_4}$ lies in $E(G)^{f_4}$. In general it is not always true that $f_i f_j = f_j f_i$ for all $i, j \in \{1, 2, \ldots, 11\}$. Thus the product fails to give a semilattice of semigroups.

The next section will give the first conclusion of this chapter.

9.4 Conclusion V-1

Let us consider again the sum in $E(S)$. Recall the endomorphism f_u which was defined in chapter 8, so that remark 8.2.2 is considered and we have a semilattice

$$\mathcal{L} = \{f_q \,; q \in \{1, 2, \ldots, 11\}\}. \tag{9.17}$$

For each pair $E(G)^{f_i}, E(G)^{f_j}$ such that $f_i \geq f_j$, we define the linking homomorphisms ϕ_{f_i,f_j} where

$$\phi_{f_i,f_j} : E(G)^{f_i} \longrightarrow E(G)^{f_j}$$

is given by

$$\textstyle\sum_r \epsilon_r \underline{\alpha_{r_i}} \longrightarrow \sum_r \epsilon_r \underline{\alpha_{r_j}} \,.$$

Hence we have a strong semilattice of groups given by

$$E(S) = (\mathcal{L}\ ,\{E(G)^{f_q}\}_{f_q \in \mathcal{L}}\ ,\ \{\phi_{f_i,f_j}\})$$

that is,

$(E(S), +)$ is a Clifford semigroup, which can be represented in the figure which is on the next page.

In the following sections we will choose a specific subset of $\text{End}(Y)$ which might give a stronger structure than the one which has been obtained. So we restart with the same proceedure as in section 9.1 but with a chosen subset of $\text{End}(Y)$.

9.5 Conclusion V-2

With the same assumptions as in section 9.1, let

$$\begin{aligned}\widehat{\text{End}}(Y) &= \{\{f_q \in \text{End}(Y)\,; f_q f_q = f_q\} \setminus \{f_{10}\}\} \subset \text{End}(Y)\\ &= \{f_1, f_2, f_3, f_4, f_6, f_7, f_8\}.\end{aligned}$$

The endomorphism f_{10} is excluded from the above set to make it closed under product.

Let $\widehat{\text{End}}(S)$ denote the set of endomorphisms of S of type f_q, where $f_q \in \widehat{\text{End}}(Y)$.

Let $\widehat{E(S)}$ be the d.g. seminear-ring generated by $\widehat{\text{End}}(S)$.

For $q \in \{1, 2, 3, 4, 6, 7, 8\}$, define the maps $\widehat{\Gamma}_q$ where

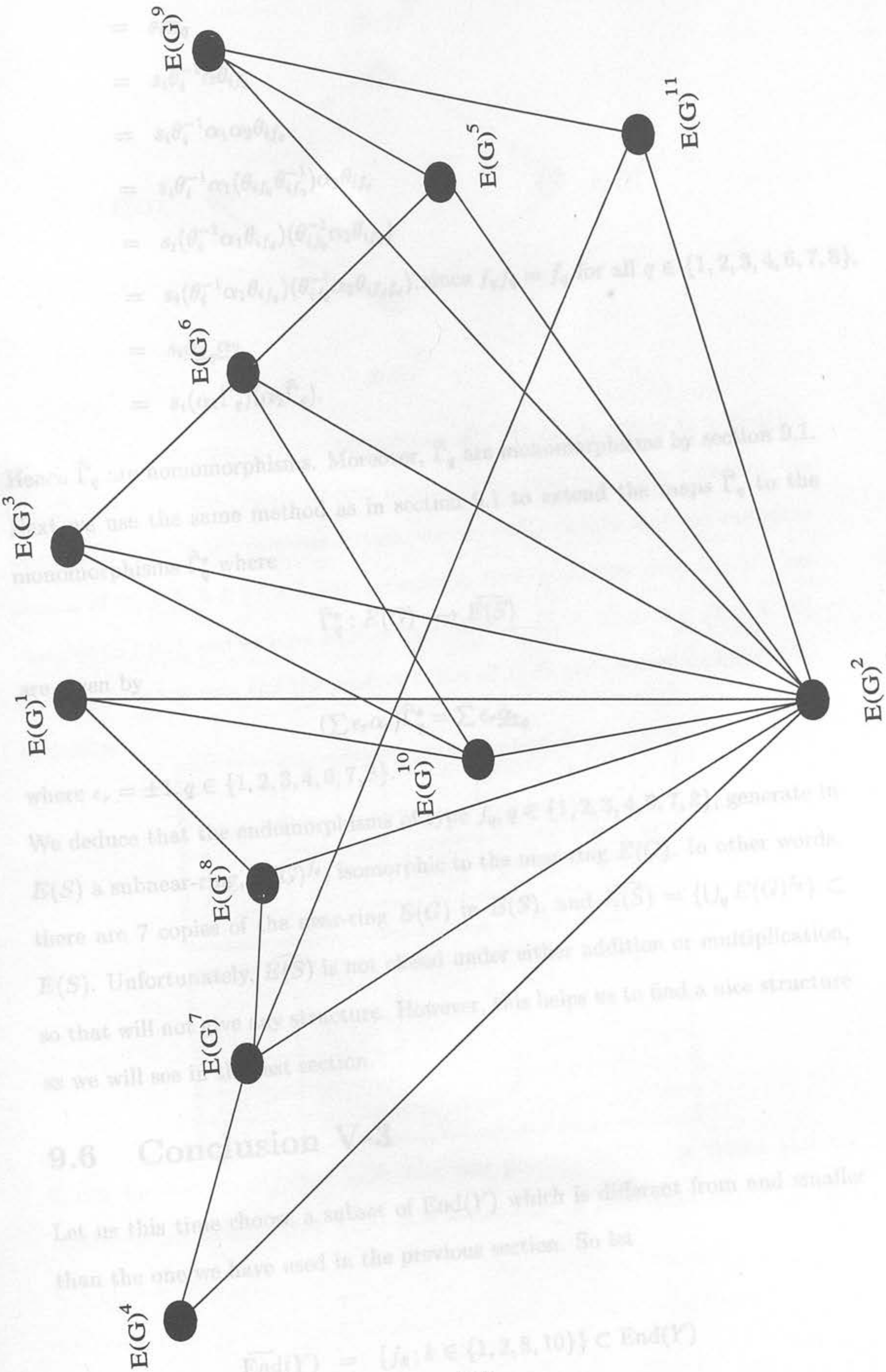
$$\widehat{\Gamma}_q : \text{End}(G) \longrightarrow \widehat{\text{End}}(S) \subset \text{End}(S)$$

are given by

$$(\alpha)\widehat{\Gamma}_q = \underline{\alpha}_q.$$

We show that the maps $\widehat{\Gamma}_q$ are homomorphisms. Let $\alpha_1, \alpha_2 \in \text{End}(G)$, and let $s_i \in S$ where $s_i \in G_i$ for some $i = 0, 1, 2$. Then

$$s_i(\alpha_1\alpha_2)\widehat{\Gamma}_q \;=\; s_i(\alpha)\widehat{\Gamma}_q, \text{where } \alpha = \alpha_1\alpha_2,$$


E(G)9
E(G)11
E(G)5
E(G)6
E(G)3
E(G)1
E(G)2
E(G)10
E(G)8
E(G)7
E(G)4

$$\begin{aligned}
&= s_i \underline{\alpha}_q \\
&= s_i \theta_i^{-1} \alpha \theta_{if_q} \\
&= s_i \theta_i^{-1} \alpha_1 \alpha_2 \theta_{if_q} \\
&= s_i \theta_i^{-1} \alpha_1 (\theta_{if_q} \theta_{if_q}^{-1}) \alpha_2 \theta_{if_q} \\
&= s_i (\theta_i^{-1} \alpha_1 \theta_{if_q})(\theta_{if_q}^{-1} \alpha_2 \theta_{if_q}) \\
&= s_i (\theta_i^{-1} \alpha_1 \theta_{if_q})(\theta_{if_q}^{-1} \alpha_2 \theta_{if_q f_q}), \text{since } f_q f_q = f_q \text{ for all } q \in \{1,2,3,4,6,7,8\}, \\
&= s_i \underline{\alpha_1}_q \underline{\alpha_2}_q \\
&= s_i (\alpha_1 \widehat{\Gamma}_q)(\alpha_2 \widehat{\Gamma}_q).
\end{aligned}$$

Hence $\widehat{\Gamma}_q$ are homomorphisms. Moreover, $\widehat{\Gamma}_q$ are monomorphisms by section 9.1. Next we use the same method as in section 9.1 to extend the maps $\widehat{\Gamma}_q$ to the monomorphisms $\widehat{\Gamma}_q^{\star}$ where

$$\widehat{\Gamma}_q^{\star} : E(G) \longrightarrow \widehat{E(S)}$$

are given by

$$(\sum \epsilon_r \alpha_r)\widehat{\Gamma}_q^{\star} = \sum \epsilon_r \underline{\alpha_r}_q$$

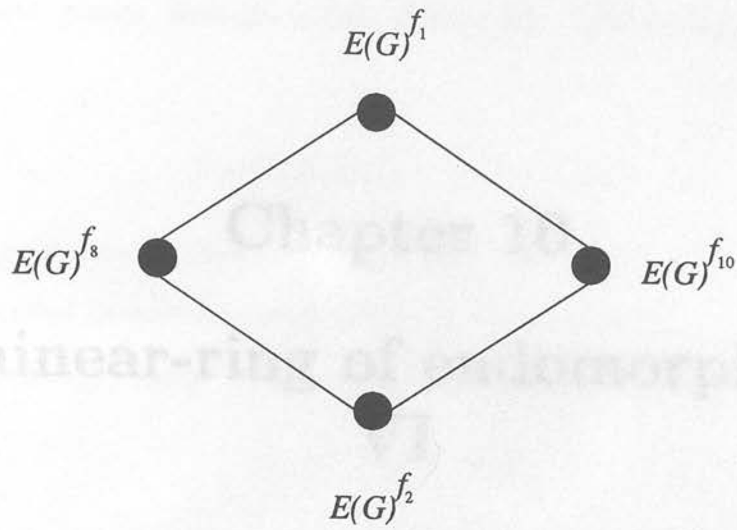
where $\epsilon_r = \pm 1, q \in \{1,2,3,4,6,7,8\}$.

We deduce that the endomorphisms of type $f_q, q \in \{1,2,3,4,6,7,8\}$, generate in $E(S)$ a subnear-ring, $E(G)^{f_q}$, isomorphic to the near-ring $E(G)$. In other words, there are 7 copies of the near-ring $E(G)$ in $E(S)$, and $\widehat{E(S)} = \{\bigcup_q E(G)^{f_q}\} \subset E(S)$. Unfortunately, $\widehat{E(S)}$ is not closed under either addition or multiplication, so that will not give any structure. However, this helps us to find a nice structure as we will see in the last section.

9.6 Conclusion V-3

Let us this time choose a subset of $\mathrm{End}(Y)$ which is different from and smaller than the one we have used in the previous section. So let

$$\widetilde{\mathrm{End}}(Y) \;=\; \{f_k \,;\, k \in \{1,2,8,10\}\} \subset \mathrm{End}(Y)$$



Let $\widetilde{\mathrm{End}}(S)$ denote the set of endomorphisms of S of type f_k, where $f_k \in \widetilde{\mathrm{End}}(Y)$. Let $\widetilde{E(S)}$ be the d.g. seminear-ring generated by $\widetilde{\mathrm{End}}(S)$. Then following the same pattern as in the previous section, we can easily deduce that the endomorphisms of type $f_k, k \in \{1, 2, 8, 10\}$, generate in $E(S)$ a subnear-ring, $E(G)^{f_k}$, isomorphic to $E(G)$, and we have $\widetilde{E(S)} = \bigcup_k E(G)^{f_k} \subset E(S)$, where $k \in \{1, 2, 8, 10\}$. Now we draw the sum and the product tables of $\widetilde{E(S)}$ and observe an interesting conlusion.

$+$	$E(G)^{f_1}$	$E(G)^{f_2}$	$E(G)^{f_8}$	$E(G)^{f_{10}}$
$E(G)^{f_1}$	$E(G)^{f_1}$	$E(G)^{f_2}$	$E(G)^{f_8}$	$E(G)^{f_{10}}$
$E(G)^{f_2}$	$E(G)^{f_2}$	$E(G)^{f_2}$	$E(G)^{f_2}$	$E(G)^{f_2}$
$E(G)^{f_8}$	$E(G)^{f_8}$	$E(G)^{f_2}$	$E(G)^{f_8}$	$E(G)^{f_2}$
$E(G)^{f_{10}}$	$E(G)^{f_{10}}$	$E(G)^{f_2}$	$E(G)^{f_2}$	$E(G)^{f_{10}}$

$\cdot$	$E(G)^{f_1}$	$E(G)^{f_2}$	$E(G)^{f_8}$	$E(G)^{f_{10}}$
$E(G)^{f_1}$	$E(G)^{f_1}$	$E(G)^{f_2}$	$E(G)^{f_8}$	$E(G)^{f_{10}}$
$E(G)^{f_2}$	$E(G)^{f_2}$	$E(G)^{f_2}$	$E(G)^{f_2}$	$E(G)^{f_2}$
$E(G)^{f_8}$	$E(G)^{f_8}$	$E(G)^{f_2}$	$E(G)^{f_8}$	$E(G)^{f_2}$
$E(G)^{f_{10}}$	$E(G)^{f_{10}}$	$E(G)^{f_2}$	$E(G)^{f_2}$	$E(G)^{f_{10}}$

It can be seen clearly that both sum and product tables are closed and coincide, which asserts that $\widetilde{E(S)}$ is a semilattice of near-rings $E(G)^{f_k}$, where $k \in \{1, 2, 8, 10\}$.

Chapter 10

Seminear-ring of endomorphisms VI

In this chapter we generalize case V in chapter 9 so that our groups are not necessarily isomorphic. However, they are linked by epimorphisms. We should point out that some details regarding the work in this chapter are omitted when it is assumed that it has appeared (or can be obtained in a similar way to that used) on some occasion in our work in chapter 5 to chapter 9. However, when this happens, we will be referring to the place it appeared.

10.1 Starting case VI

Let

$$Y = \{0, 1, 2\}$$

be a semilattice with $1 \geq 0$ and $2 \geq 0$ (1 and 2 are not comparable).

Consider the groups G_0, G_1 and G_2 which have the linking epimorphisms $\phi_{1,0}$ and $\phi_{2,0}$ such that

$$\phi_{1,0} : G_1 \longrightarrow G_0$$

and

$$\phi_{2,0} : G_2 \longrightarrow G_0.$$

Consider $S = \bigcup_{i=0,1,2} G_i$. As in the previous cases, by theorem 4.2.3, for an endomorphism f of $S, f|_Y$ is an endomorphism of Y, and if we restrict f on any of the above groups, then the situation is known. Furthermore, by case V, we have

$$\mathrm{End}(Y) = \{f_q\,;q \in \{1, 2, \ldots, 11\}\}$$

where those endomorphisms are given on page 118. As before, an endomorphism f of S is called an endomorphism of type f_q when $f|_Y = f_q$.
Let $\mathrm{K}_1 = \mathrm{Ker}\phi_{1,0}$ and $\mathrm{K}_2 = \mathrm{Ker}\phi_{2,0}$. Then

$$G_1/\mathrm{K}_1 \cong G_0 \cong G_2/\mathrm{K}_2.$$

Suppose that $f \in \mathrm{End}(S)$ such that $f|_{G_1} \in \mathrm{End}(G_1)$. Then, as in case II in chapter 6, we can show that

$$\mathrm{K}_1 f \subseteq \mathrm{K}_1 \tag{10.1}$$

Similarly when $f|_{G_2} \in \mathrm{End}(G_2)$, we get

$$\mathrm{K}_2 f \subseteq \mathrm{K}_2. \tag{10.2}$$

Consider the endomorphisms of type f_1 on S, then there exist maps α, β, and ω such that

$$f|_{G_2} = \omega \in \mathrm{End}(G_2),\ f|_{G_1} = \alpha \in \mathrm{End}(G_1),\ f|_{G_0} = \beta \in \mathrm{End}(G_0).$$

From above, we have $\mathrm{K}_1\alpha \subseteq \mathrm{K}_1$ and $\mathrm{K}_2\omega \subseteq \mathrm{K}_2$. Following the same pattern as in chapter 6, we can obtain the following two equations :

$$\alpha\phi_{1,0} = \phi_{1,0}\beta \tag{10.3}$$

$$\omega\phi_{2,0} = \phi_{2,0}\beta. \tag{10.4}$$

Again by the same method applied in chapter 6, we can define endomorphisms $\bar{\alpha}$ and $\hat{\omega}$ such that

$$\bar{\alpha} : G_1/\mathrm{K}_1 \longrightarrow G_1/\mathrm{K}_1$$

is given by

$$(g_1 + \mathrm{K}_1)\bar{\alpha} = g_1\alpha + \mathrm{K}_1$$

and

$$\hat{\omega} : G_2/\mathrm{K}_2 \longrightarrow G_2/\mathrm{K}_2$$

is given by

$$(g_2 + \mathrm{K}_2)\hat{\omega} = g_2\omega + \mathrm{K}_2.$$

Since $G_1/\mathrm{K}_1 \cong G_0 \cong G_2/\mathrm{K}_2$, $\bar{\alpha}$ and $\hat{\omega}$ could be considered as endomorphisms of G_0. Let Δ be the isomorphism mapping G_1/K_1 to G_0 where

$$\Delta : G_1/\mathrm{K}_1 \longrightarrow G_0$$

is given by

$$(g_1 + \mathrm{K})\Delta = g_1\phi_{1,0}$$

and let ψ be the isomorphism mapping $\mathrm{End}(G_1/\mathrm{K}_1)$ to $\mathrm{End}(G_0)$.

Then the isomorphism

$$\psi : \mathrm{End}(G_1/\mathrm{K}_1) \longrightarrow \mathrm{End}(G_0)$$

is defined by

$$\bar{\alpha}\psi = \Delta^{-1}\bar{\alpha}\Delta.$$

A similar situation to case II, we deduce that

$$\bar{\alpha}\psi \;=\; \beta \tag{10.5}$$

and we can write (10.3) as

$$\alpha\phi_{1,0} \;=\; \phi_{1,0}\bar{\alpha}\psi. \tag{10.6}$$

Similarly, we consider the isomorphism

$$\eta : G_2/\mathrm{K}_2 \longrightarrow G_0$$

which is given by

$$(g_2 + \mathrm{K}_2)\eta = g_2\phi_{2,0}$$

then the isomorphism

$$\chi : \mathrm{End}(G_2/\mathrm{K}_2) \longrightarrow \mathrm{End}(G_0)$$

is defined by

$$\hat{\omega}\chi = \eta^{-1}\hat{\omega}\eta$$

and we have a similar situation to (10.5), which is

$$\hat{\omega}\chi \;=\; \beta \tag{10.7}$$

so we have

$$\bar{\alpha}\psi \;=\; \beta = \hat{\omega}\chi. \tag{10.8}$$

Hence given $\alpha \in \mathrm{End}(G_1)$ and $\omega \in \mathrm{End}(G_2)$ such that $\mathrm{K}_1\alpha \subseteq \mathrm{K}_1$, $\mathrm{K}_2\omega \subseteq \mathrm{K}_2$ and $\bar{\alpha}\psi = \hat{\omega}\chi$, we can define a map

$$\underline{\alpha, \omega}_1 : S \longrightarrow S$$

$$(s)\underline{\alpha, \omega}_1 = \begin{cases} s\omega & \text{if } s \in G_2 , \\ s\alpha & \text{if } s \in G_1 , \\ s\bar{\alpha}\psi & \text{if } s \in G_0. \end{cases}$$

Then $\underline{\alpha, \omega}_1$ is an endomorphism of type f_1 on S.

Consider now the endomorphisms of type f_2 on S, then for such an endomorphism f there exist σ, δ and β such that

$$f|_{G_2} = \sigma \in \mathrm{Hom}(G_2, G_0),\ f|_{G_1} = \delta \in \mathrm{Hom}(G_1, G_0),\ f|_{G_0} = \beta \in \mathrm{End}(G_0).$$

By theorem 4.2.4, we have

$$G_2\phi_{2,0}f = G_2 f$$

which implies that

$$\phi_{2,0}\beta \;=\; \sigma \tag{10.9}$$

also

$$G_1\phi_{1,0}f \;=\; G_1 f$$

which implies that

$$\phi_{1,0}\beta \;=\; \delta. \tag{10.10}$$

So given $\beta \in \mathrm{End}(G_0)$, we can define a map

$$\underline{\beta}_2 : S \longrightarrow S$$

$$(s)\underline{\beta}_2 = \begin{cases} s\phi_{2,0}\beta & \text{if } s \in G_2 \ , \\ s\phi_{1,0}\beta & \text{if } s \in G_1 \ , \\ s\beta & \text{if } s \in G_0. \end{cases}$$

Then the map $\underline{\beta}_2$ is an endomorphism of type f_2 on S.

Next we consider the endomorphisms of type f_3 on S. For such an endomorphism f there exist μ, α and γ such that

$$f|_{G_2} = \mu \in \mathrm{Hom}(G_2, G_1), \ f|_{G_1} = \alpha \in \mathrm{End}(G_1), \ f|_{G_0} = \gamma \in \mathrm{Hom}(G_0, G_1).$$

By theorem 4.2.4, we have

$$G_2\phi_{2,0}f = G_2f$$

which implies that

$$\phi_{2,0}\gamma = \mu \tag{10.11}$$

also

$$G_1\phi_{1,0}f = G_1f$$

which gives

$$\phi_{1,0}\gamma = \alpha. \tag{10.12}$$

It should be noticed that proposition 6.2.1 is valid in this case and we have

$$(\overline{\phi_{1,0}\gamma})\psi = \gamma\phi_{1,0}. \tag{10.13}$$

However for a given $\gamma \in \mathrm{Hom}(G_0, G_1)$, we define a map

$$\underline{\gamma}_3 : S \longrightarrow S$$

$$(s)\underline{\gamma}_3 = \begin{cases} s\phi_{2,0}\gamma & \text{if } s \in G_2 \ , \\ s\phi_{1,0}\gamma & \text{if } s \in G_1 \ , \\ s\gamma & \text{if } s \in G_0. \end{cases}$$

The map $\underline{\gamma}_3$ is an endomorphism of type f_3 on S.

Now we consider the endomorphisms of type f_4 on S. For such an endomorphism f there exist ω, λ and ρ such that

$$f|_{G_2} = \omega \in \mathrm{End}(G_2),\ f|_{G_1} = \lambda \in \mathrm{Hom}(G_1, G_2),\ f|_{G_0} = \rho \in \mathrm{Hom}(G_0, G_2).$$

By theorem 4.2.4, we have

$$G_2\phi_{2,0}f \;=\; G_2 f$$

which implies that

$$\phi_{2,0}\rho \;=\; \omega \tag{10.14}$$

also

$$G_1\phi_{1,0}f \;=\; G_1 f$$

which shows that

$$\phi_{1,0}\rho = \lambda. \tag{10.15}$$

Hence for a given $\rho \in \mathrm{Hom}(G_0, G_2)$, we define a map

$$\underline{\rho}_4 : S \longrightarrow S$$

$$(s)\underline{\rho}_4 = \begin{cases} s\phi_{2,0}\rho & \text{if } s \in G_2\ , \\ s\phi_{1,0}\rho & \text{if } s \in G_1\ , \\ s\rho & \text{if } s \in G_0. \end{cases}$$

The map $\underline{\rho}_4$ is an endomorphism of type f_4 on S.

Consider now the endomorphisms of type f_5 on S. For such an endomorphism f there exist μ, δ and β such that

$$f|_{G_2} = \mu \in \mathrm{Hom}(G_2, G_1),\ f|_{G_1} = \delta \in \mathrm{Hom}(G_1, G_0),\ f|_{G_0} = \beta \in \mathrm{End}(G_0).$$

By theorem 4.2.4, we have

$$G_2\phi_{2,0}f \;=\; G_2 f\phi_{1,0}$$

which implies that

$$\phi_{2,0}\beta \;=\; \mu\phi_{1,0} \tag{10.16}$$

also we have

$$G_1\phi_{1,0}f \;=\; G_1 f$$

$$\underline{\alpha, \mu}_6 : S \longrightarrow S$$

$$(s)\underline{\alpha, \mu}_6 = \begin{cases} s\mu & \text{if } s \in G_2 , \\ s\alpha & \text{if } s \in G_1 , \\ s\bar{\alpha}\psi & \text{if } s \in G_0. \end{cases}$$

The map $\underline{\alpha, \mu}_6$ is an endomorphism of type f_6 on S.

Next we consider the endomorphisms of type f_7 on S. For such an endomorphism f there exist ω, λ and β such that

$$f|_{G_2} = \omega \in \text{End}(G_2),\ f|_{G_1} = \lambda \in \text{Hom}(G_1, G_2),\ f|_{G_0} = \beta \in \text{End}(G_0).$$

Let $\mathrm{K}_2 = \text{Ker}\phi_{2,0}$, then $\mathrm{K}_2\omega \subseteq \mathrm{K}_2$. Furthermore, equation (10.7), with the same assumption, is satisfied and by theorem 4.2.4, we have,

$$\phi_{2,0}\beta \;=\; \omega\phi_{2,0} \tag{10.22}$$

and

$$\phi_{1,0}\beta \;=\; \lambda\phi_{2,0} \tag{10.23}$$

which could be written as

$$\phi_{1,0}\hat{\omega}\chi \;=\; \lambda\phi_{2,0}. \tag{10.24}$$

Hence, given $\omega \in \text{End}(G_2)$ and $\lambda \in \text{Hom}(G_1, G_2)$ such that $\mathrm{K}_2\omega \subseteq \mathrm{K}_2$ and equation (10.24) is satisfied, we can define a map

$$\underline{\omega, \lambda}_7 : S \longrightarrow S$$

$$(s)\underline{\omega, \lambda}_7 = \begin{cases} s\omega & \text{if } s \in G_2 , \\ s\lambda & \text{if } s \in G_1 , \\ s\hat{\omega}\chi & \text{if } s \in G_0. \end{cases}$$

The map $\underline{\omega, \lambda}_7$ is an endomorphism of type f_7 on S.

Consider now the endomorphisms of type f_8 on S. For such an endomorphism f there exist ω, δ and β such that

$$f|_{G_2} = \omega \in \text{End}(G_2),\ f|_{G_1} = \delta \in \text{Hom}(G_1, G_0),\ f|_{G_0} = \beta \in \text{End}(G_0).$$

Again in this case we can observe that $K_2\omega \subseteq K_2$ where $K_2 = \mathrm{Ker}\phi_{2,0}$ and equations (10.7), (10.17) and (10.22) are satisfied and we can write (10.17) as

$$\phi_{1,0}\hat{\omega}\chi \;=\; \delta. \tag{10.25}$$

Hence, given $\omega \in \mathrm{End}(G_2)$ such that $K_2\omega \subseteq K_2$, we define a map

$$\underline{\omega_8} : S \longrightarrow S$$

$$(s)\underline{\omega_8} = \begin{cases} s\omega & \text{if } s \in G_2\ , \\ s\phi_{1,0}\hat{\omega}\chi & \text{if } s \in G_1\ , \\ s\hat{\omega}\chi & \text{if } s \in G_0. \end{cases}$$

The map $\underline{\omega_8}$ is an endomorphism of type f_8 on S.

Next we consider the endomorphisms of type f_9 on S. For such an endomorphism f there exist μ, λ and β such that

$$f|_{G_2} = \mu \in \mathrm{Hom}(G_2, G_1),\ f|_{G_1} = \lambda \in \mathrm{Hom}(G_1, G_2),\ f|_{G_0} = \beta \in \mathrm{End}(G_0).$$

By theorem 4.2.4, we have,

$$\phi_{2,0}\beta \;=\; \mu\phi_{1,0} \tag{10.26}$$

and

$$\phi_{1,0}\beta \;=\; \lambda\phi_{2,0}. \tag{10.27}$$

Hence, given $\beta \in \mathrm{End}(G_0)$, $\lambda \in \mathrm{Hom}(G_1, G_2)$ and $\mu \in \mathrm{Hom}(G_2, G_1)$ such that both equations (10.26) and (10.27) are satisfied, we can define a map

$$\underline{\beta, \lambda, \mu}_9 : S \longrightarrow S$$

$$(s)\underline{\beta, \lambda, \mu}_9 = \begin{cases} s\mu & \text{if } s \in G_2\ , \\ s\lambda & \text{if } s \in G_1\ , \\ s\beta & \text{if } s \in G_0. \end{cases}$$

The map $\underline{\beta, \lambda, \mu}_9$ is an endomorphism of type f_9 on S.

Consider now the endomorphisms of type f_{10} on S. For such an endomorphism f, there exist σ, α and β such that

$$f|_{G_2} = \sigma \in \mathrm{Hom}(G_2, G_0),\ f|_{G_1} = \alpha \in \mathrm{End}(G_1),\ f|_{G_0} = \beta \in \mathrm{End}(G_0).$$

As in the situation of type f_1, f satisfies $\mathrm{K}_1\alpha \subseteq \mathrm{K}_1$ where $\mathrm{K}_1 = \mathrm{Ker}\phi_{1,0}$ and equations (10.3) and (10.6) are satisfied. Thus, for a given $\alpha \in \mathrm{End}(G_1)$ we define a map

$$\underline{\alpha}_{10} : S \longrightarrow S$$

$$(s)\underline{\alpha}_{10} = \begin{cases} s\phi_{2,0}\bar{\alpha}\psi & \text{if } s \in G_2\ , \\ s\alpha & \text{if } s \in G_1, \\ s\bar{\alpha}\psi & \text{if } s \in G_0. \end{cases}$$

Then $\underline{\alpha}_{10}$ is an endomorphism of type f_{10} on S.

Finally we consider the endomorphisms of type f_{11} on S. For such an endomorphism f there exist σ, λ and β such that

$$f|_{G_2} = \sigma \in \mathrm{Hom}(G_2, G_0),\ f|_{G_1} = \lambda \in \mathrm{Hom}(G_1, G_2),\ f|_{G_0} = \beta \in \mathrm{End}(G_0).$$

It is now easy to see that equations (10.9) and (10.23) are satisfied. However, given $\beta \in \mathrm{End}(G_0)$ and $\lambda \in \mathrm{Hom}(G_1, G_2)$ such that equation (10.23) is satisfied we can define a map

$$\underline{\beta, \lambda}_{11} : S \longrightarrow S$$

$$(s)\underline{\beta, \lambda}_{11} = \begin{cases} s\phi_{2,0}\beta & \text{if } s \in G_2\ , \\ s\lambda & \text{if } s \in G_1\ , \\ s\beta & \text{if } s \in G_0. \end{cases}$$

The map $\underline{\beta, \lambda}_{11}$ is an endomorphism of type f_{11} on S.

Now we are going to consider each type of endomorphism (11 types) separately, and we will be following a similar scheme in each one. Our plan is as follows : we start with a type of endomorphism and construct a near-ring or a group. Then we link up our structure with $E(S)$ relating to the underlying type of endomorphism. Then we will observe that the near-ring generated in $E(S)$ by this type of endomorphism is isomorphic to the structure we already started with, and so on. As we have mentioned above, we will be repeating these steps for all the types of endomorphism which we have, and therefore we are not going to give all the details in each step. However, some of those are chosen to be written in detail.

We start by letting

$$M_1 : = \{\alpha \in \mathrm{End}(G_1); \mathrm{K}_1\alpha \subseteq \mathrm{K}_1 \text{ and } \bar{\alpha}\psi = \hat{\omega}\chi \text{ for some } \omega \in \mathrm{End}(G_2)\}$$

$$M_2 : = \{\omega \in \mathrm{End}(G_2); \mathrm{K}_2\omega \subseteq \mathrm{K}_2 \text{ and } \hat{\omega}\chi = \bar{\alpha}\psi \text{ for some } \alpha \in \mathrm{End}(G_1)\}$$

$$M : = \{\beta \in \mathrm{End}(G_0); \exists\, \alpha \in M_1 \text{ and } \exists\, \omega \in M_2 \text{ such that } \bar{\alpha}\psi = \beta = \hat{\omega}\chi\}.$$

Thus, we have three near-rings, namely, N_1 , N_2 and N_3 where

$$\mathrm{N}_1 = \mathrm{Nr}<\mathrm{M}_1> \subseteq \mathrm{E}(G_1)$$

$$\mathrm{N}_2 = \mathrm{Nr}<\mathrm{M}_2> \subseteq \mathrm{E}(G_2)$$

$$\mathrm{N} = \mathrm{Nr}<\mathrm{M}> \subseteq \mathrm{E}(G_0).$$

From above, there exist epimorphisms τ_1 and τ_2 such that

$$\tau_1 : \mathrm{M}_1 \longrightarrow \mathrm{M}$$

is defined by

$$(\alpha)\tau_1 = \bar{\alpha}\psi$$

and

$$\tau_2 : \mathrm{M}_2 \longrightarrow \mathrm{M}$$

is defined by

$$(\omega)\tau_2 = \hat{\omega}\chi.$$

We now extend the homomorphisms τ_1 and τ_2 to $\tau_1^\star$ and $\tau_2^\star$ respectively, where

$$\tau_1^\star : \mathrm{N}_1 \longrightarrow \mathrm{N}$$

is given by

$$(\sum \epsilon_i\alpha_i)\tau_1^\star = \sum \epsilon_i(\bar{\alpha}_i\psi)$$

where $\epsilon_i = \pm 1$, for all i, and

$$\tau_2^\star : \mathrm{N}_2 \longrightarrow \mathrm{N}$$

is given by

$$(\sum \eta_i\omega_j)\tau_2^\star = \sum \eta_j(\hat{\omega}_j\chi)$$

where $\eta_j = \pm 1$, for all j.

We show that the maps $\tau_1^\star$ and $\tau_2^\star$ are homomorphisms. Consider $\tau_1^\star$ and suppose that $c \in \mathrm{N}_1$ such that $c = \sum_{i=1}^{n} \epsilon_i \alpha_i = 0$. We show that $(c)\tau_1^\star = 0$. By assumption,

$$g_1 \sum_{i=1}^{n} \epsilon_i \alpha_i = 0, \text{for all } g_1 \in G_1.$$

So for $g_0 \in G_0$, we have

$$\begin{aligned} g_0(c)\tau_1^\star &= g_0(\sum_{i=1}^{n} \epsilon_i \alpha_i)\tau_1^\star \\ &= g_0(\sum_{i=1}^{n} \epsilon_i \bar{\alpha}_i \psi) \\ &= g_1 \phi_{1,0}(\sum_{i=1}^{n} \epsilon_i \bar{\alpha}_i \psi) \text{ , where } g_1\phi_{1,0} = g_0 \text{ ,} \\ &= g_1(\sum_{i=1}^{n} \epsilon_i \phi_{1,0} \bar{\alpha}_i \psi) \\ &= g_1(\sum_{i=1}^{n} \epsilon_i \alpha_i \phi_{1,0}) \text{ , by (10.6) ,} \\ &= 0. \end{aligned}$$

Hence $\tau_1^\star$ is a homomorphism.

Similarly, we can show that the map $\tau_2^\star$ is a homomorphism.

Let

$$\mathrm{P}_1 := \{(\alpha, \omega) \subseteq \mathrm{M}_1 \times \mathrm{M}_2; \alpha\tau_1 = \omega\tau_2\}.$$

We can see that the set P_1 is closed under products, for if $x, y \in \mathrm{P}_1$, where $x = (\alpha, \omega)$ and $y = (\alpha', \omega')$, then $xy = (\alpha\alpha', \omega\omega')$. Since $\alpha\tau_1 = \omega\tau_2$ and $\alpha'\tau_1 = \omega'\tau_2$, it follows that $\overline{\alpha}\psi = \hat{\omega}\chi$ and $\overline{\alpha'}\psi = \hat{\omega'}\chi$.

So

$$(\overline{\alpha\alpha'})\psi = (\overline{\alpha}\psi)(\overline{\alpha'}\psi) = (\hat{\omega}\chi)(\hat{\omega'}\chi) = (\hat{\omega\omega'}\chi)$$

which implies that

$$(\alpha\alpha')\tau_1 = (\omega\omega')\tau_2.$$

Hence $xy \in \mathrm{P}_1$ and P_1 is closed under products.

Now we link up the set P_1 with $\mathrm{End}(S)$ by defining a map

$$\Gamma_1 : \mathrm{P}_1 \longrightarrow \mathrm{End}(S)$$

by

$$(\alpha, \omega)\Gamma_1 = \underline{\alpha, \omega}_1.$$

We show that the map Γ_1 is a homomorphism. Let $x, y \in \mathrm{P}$, where $x = (\alpha, \omega), y = (\alpha', \omega')$. Then

$$(xy)\Gamma_1 = ((\alpha, \omega)(\alpha', \omega'))\Gamma_1 \;=\; (\alpha\alpha', \omega\omega')\Gamma_1 = \underline{\alpha\alpha', \omega\omega'}_1$$

So for $s_i \in G_i$ where $i = 0, 1, 2$, we have

$$s_2(xy)\Gamma_1 \;=\; (s_2)\omega\omega' \tag{10.28}$$

$$s_1(xy)\Gamma_1 \;=\; (s_1)\alpha\alpha' \tag{10.29}$$

$$s_0(xy)\Gamma_1 \;=\; (s_0)\overline{\alpha\alpha'}\psi. \tag{10.30}$$

On the other hand, since $(x)\Gamma_1(y)\Gamma_1 = (\alpha, \omega)\Gamma_1(\alpha', \omega')\Gamma_1 = \underline{\alpha, \omega}_1\underline{\alpha', \omega'}_1$, we have

$$s_2(x)\Gamma_1(y)\Gamma_1 \;=\; s_2\underline{\alpha, \omega}_1\underline{\alpha', \omega'}_1 = (s_2\omega)\underline{\alpha', \omega'}_1 = s_2\omega\omega' \tag{10.31}$$

$$s_1(x)\Gamma_1(y)\Gamma_1 \;=\; s_1\underline{\alpha, \omega}_1\underline{\alpha', \omega'}_1 = (s_1\alpha)\underline{\alpha', \omega'}_1 = s_1\alpha\alpha' \tag{10.32}$$

$$\begin{aligned} s_0(x)\Gamma_1(y)\Gamma_1 \;&=\; s_0\underline{\alpha, \omega}_1\underline{\alpha', \omega'}_1 = (s_0\overline{\alpha}\psi)\underline{\alpha', \omega'}_1 \\ &=\; (s_0)\overline{\alpha}\psi\overline{\alpha'}\psi = (s_0)\overline{\alpha\alpha'}\psi. \end{aligned} \tag{10.33}$$

Equations (10.28) — (10.33) show that $(xy)\Gamma_1 = (x)\Gamma_1(y)\Gamma_1$ and Γ_1 is a homomorphism.

Let

$$\overline{\mathrm{P}_1} := \mathrm{Snr}<\mathrm{P}_1> \subseteq \mathrm{N}_1 \times \mathrm{N}_2.$$

Then we can extend the map Γ_1 to $\Gamma_1^\star$ such that

$$\Gamma_1^\star : \overline{\mathrm{P}_1} \longrightarrow E(S)$$

is defined by

$$(a, b)\Gamma_1^\star = \textstyle\sum_{i=1}^n \epsilon_i \underline{\alpha_i, \omega_i}_1$$

where, $a = \sum_{i=1}^{n} \epsilon_i \alpha_i, b = \sum_{i=1}^{n} \epsilon_i \omega_i, \epsilon_i = \pm 1$.

We show that the map $\Gamma_1^\star$ is a homomorphism. Suppose that $0 = (a, b) \in \overline{\mathrm{P}_1}$, where $a = \sum_{i=1}^{n} \epsilon_i \alpha_i$ and $b = \sum_{i=1}^{n} \epsilon_i \omega_i$, then

$$g_2 \sum_{i=1}^{n} \epsilon_i \omega_i = 0 \text{ and } g_1 \sum_{i=1}^{n} \epsilon_i \alpha_i = 0 \text{ for all } g_2 \in G_2, g_1 \in G_1,$$

which also give

$$g_0 \sum_{i=1}^{n} \epsilon_i \bar{\alpha}_i \psi = \sum_{i=1}^{n} \epsilon_i g_0 \bar{\alpha}_i \psi = 0 \text{ for all } g_0 \in G_0.$$

Similarly $g_0 \sum_{i=1}^{n} \epsilon_i \hat{\omega}_i \chi = 0$. Hence, $s \sum_{i=1}^{n} \epsilon_i \underline{\alpha_i, \omega_i}_1 = 0$ for all $s \in S$, which means that $(a, b)\Gamma_1^\star = 0$ and $\Gamma_1^\star$ is a homomorphism.

Next we show that $\Gamma_1^\star$ is a monomorphism. Suppose that $0 \neq (a, b) \in \overline{\mathrm{P}_1}$, where $a = \sum_{i=1}^{n} \epsilon_i \alpha_i$ and $b = \sum_{i=1}^{n} \epsilon_i \omega_i$; then either $a \neq 0$ or $b \neq 0$.

If $a \neq 0$, then there exists $g_1 \in G_1$ such that $g_1 \sum_{i=1}^{n} \epsilon_i \alpha_i \neq 0$, which will imply that $g_1 \sum_{i=1}^{n} \epsilon_i \underline{\alpha_i, \omega_i}_1 \neq 0$ and $(a, b)\Gamma_1^\star \neq 0$.

If $b \neq 0$, then there exists $g_2 \in G_2$ such that $g_2 \sum_{i=1}^{n} \epsilon_i \omega_i \neq 0$, which will imply that $g_2 \sum_{i=1}^{n} \epsilon_i \underline{\alpha_i, \omega_i}_1 \neq 0$ and $(a, b)\Gamma_1^\star \neq 0$. Hence $\Gamma_1^\star$ is a monomorphism.

We deduce that the endomorphisms of type f_1 on S generate in $E(S)$ a subnear-ring, which we denote by E^{f_1}, isomorphic to $\overline{\mathrm{P}_1}$, where $\overline{\mathrm{P}_1}$, as defined earlier, is a subnear-ring of $\mathrm{N}_1 \oplus \mathrm{N}_2$. Recall that the projection maps π_1 and π_2 are onto, where

$$\pi_1 : \overline{\mathrm{P}_1} \longrightarrow \mathrm{N}_1$$

is defined by

$$\left(\sum_{i=1}^{n} \epsilon_i (\alpha_i, \omega_i)\right)\pi_1 = \sum_{i=1}^{n} \epsilon_i \alpha_i$$

and

$$\pi_2 : \overline{\mathrm{P}_1} \longrightarrow \mathrm{N}_2$$

is defined by

$$\left(\sum_{i=1}^{n} \epsilon_i (\alpha_i, \omega_i)\right)\pi_2 = \sum_{i=1}^{n} \epsilon_i \omega_i.$$

Hence E^{f_1} is a subdirect product of $\mathrm{N}_1 \oplus \mathrm{N}_2$.

Next we consider $\mathrm{End}(G_0)$ to be linked with the seminear-ring $E(S)$ by a map

$$\Gamma_2 : \mathrm{End}(G_0) \longrightarrow \mathrm{End}(S)$$

which is defined by

$$(\beta)\Gamma_2 = \underline{\beta}_2.$$

We show that the map Γ_2 is a homomorphism. Suppose that $\beta, \xi \in \mathrm{End}(G_0)$ and let $s_i \in G_i$, where $i = 0, 1, 2$, then

$$\begin{aligned}
s_2(\beta\xi)\Gamma_2 &= s_2(\nu)\Gamma_2, \text{where } \nu = \beta\xi, \\
&= s_2\underline{\nu}_2 \\
&= s_2\phi_{2,0}\nu \\
&= s_2\phi_{2,0}\beta\xi \\
&= s_2\underline{\beta}_2\xi \\
&= s_2\underline{\beta}_2\underline{\xi}_2 \\
&= s_2(\beta\Gamma_2)(\xi\Gamma_2).
\end{aligned} \tag{10.34}$$

$$\begin{aligned}
s_1(\beta\xi)\Gamma_2 &= s_1(\nu)\Gamma_2, \text{where } \nu = \beta\xi, \\
&= s_1\underline{\nu}_2 \\
&= s_1\phi_{1,0}\nu \\
&= s_1\phi_{1,0}\beta\xi \\
&= s_1\underline{\beta}_2\xi \\
&= s_1\underline{\beta}_2\underline{\xi}_2 \\
&= s_1(\beta\Gamma_2)(\xi\Gamma_2).
\end{aligned} \tag{10.35}$$

$$\begin{aligned}
s_0(\beta\xi)\Gamma_2 &= s_0(\nu)\Gamma_2, \text{where } \nu = \beta\xi, \\
&= s_0\underline{\nu}_2 \\
&= s_0\nu \\
&= s_0\beta\xi \\
&= s_0\underline{\beta}_2\xi \\
&= s_0\underline{\beta}_2\underline{\xi}_2 \\
&= s_0(\beta\Gamma_2)(\xi\Gamma_2).
\end{aligned} \tag{10.36}$$

Equations (10.34) — (10.36) show that $(\beta\xi)\Gamma_2 = (\beta\Gamma_2)(\xi\Gamma_2)$ and Γ_2 is a homomorphism. Now we extend the map Γ_2 to $\Gamma_2^\star$ where

$$\Gamma_2^\star : E(G_0) \longrightarrow E(S)$$

is defined by

$$(\sum \epsilon_i\beta_i)\Gamma_2^\star = \sum \epsilon_i\underline{\beta_{i_2}}$$

where $\epsilon_i = \pm 1$.

The map $\Gamma_2^\star$ is a homomorphism, for suppose that $0 = c = \sum_{i=1}^n \epsilon_i\beta_i \in E(G_0)$, then we show that $(c)\Gamma_2^\star = 0$. By assumption, $g_0 \sum_{i=1}^n \epsilon_i\beta_i = 0$ for all $g_0 \in G_0$.
Let $s_2 \in G_2$, then we have

$$\begin{aligned} s_2(c)\Gamma_2^\star &= s_2(\sum_{i=1}^n \epsilon_i\beta_i)\Gamma_2^\star \\ &= s_2(\sum_{i=1}^n \epsilon_i\underline{\beta_{i_2}}) \\ &= \sum_{i=1}^n \epsilon_i s_2\underline{\beta_{i_2}} \\ &= \sum_{i=1}^n \epsilon_i(s_2\phi_{2,0}\beta_i) \\ &= \sum_{i=1}^n \epsilon_i(s_0\beta_i), \text{where } s_0 = s_2\phi_{2,0}, \\ &= s_0\sum_{i=1}^n \epsilon_i\beta_i \\ &= 0. \end{aligned} \tag{10.37}$$

If we replace s_2 by s_1 or s_0 then we can get the same conclusion as in (10.37). Thus $(c)\Gamma_2^\star = 0$ and $\Gamma_2^\star$ is a homomorphism.

Moreover, $\Gamma_2^\star$ is a monomorphism, since if $c_1, c_2 \in E(G_0)$ such that $c_1 \neq c_2$, where $c_1 = \sum_{i=1}^n \epsilon_i\beta_i$ and $c_2 = \sum_{j=1}^r \eta_j\nu_j$, then there exists $g_0 \in G_0$ such that

$$\begin{aligned} g_0\sum_{i=1}^n \epsilon_i\beta_i &\neq g_0\sum_{j=1}^r \eta_j\nu_j \\ \sum_{i=1}^n \epsilon_i g_0\beta_i &\neq \sum_{j=1}^r \eta_j g_0\nu_j \\ \sum_{i=1}^n \epsilon_i g_0\underline{\beta_{i_2}} &\neq \sum_{j=1}^r \eta_j g_0\underline{\nu_{j_2}} \end{aligned}$$

$$
\begin{aligned}
g_0\sum_{i=1}^{n}\epsilon_i\underline{\beta_{i}}_2 &\neq g_0\sum_{j=1}^{r}\eta_j\underline{\nu_{j}}_2 \\
g_0(\sum_{i=1}^{n}\epsilon_i\beta_i)\Gamma_2^\star &\neq g_0(\sum_{j=1}^{r}\eta_j\nu_j)\Gamma_2^\star \\
(c_1)\Gamma_2^\star &\neq (c_2)\Gamma_2^\star.
\end{aligned}
$$

Hence $\Gamma_2^\star$ is a monomorphism.

We deduce that the endomorphisms of type f_2 on S generate in $E(S)$ a subnear-ring isomorphic to the near-ring $E(G_0)$.

Next we consider the set $\mathrm{Hom}(G_0, G_1)$. Following the same pattern which was applied to the endomorphisms of type II in chapter 6; we can consider the group $(\mathrm{gp}< \mathrm{Hom}(G_0, G_1) >, +)$ and obtain an isomorphic copy of this group inside $E(S)$. In this case the connection is carried out via the endomorphisms of type f_3 and the obvious map which should be considered is $\Gamma_3^\star$ where

$$\Gamma_3^\star : (\mathrm{gp}< \mathrm{Hom}(G_0, G_1) >, +) \longrightarrow E(S)$$

is given by

$$(\sum \epsilon_i\gamma_i)\Gamma_3^\star = \sum \epsilon_i\underline{\gamma_{i}}_3$$

where $\epsilon_i = \pm 1$.

If we replace the group G_1 by G_2 in the above case so that we consider the set $\mathrm{Hom}(G_0, G_2)$, then a similar argument can be done to observe an isomorphic copy of the group $(\mathrm{gp}< \mathrm{Hom}(G_0, G_2) >, +)$ inside $E(S)$. The connection here is carried out via the endomorphisms of type f_4 and in this case we consider the map $\Gamma_4^\star$ where

$$\Gamma_4^\star : (\mathrm{gp}< \mathrm{Hom}(G_0, G_2) >, +) \longrightarrow E(S)$$

is given by

$$(\sum \epsilon_i\rho_i)\Gamma_4^\star = \sum \epsilon_i\underline{\rho_{i}}_4$$

where $\epsilon_i = \pm 1$.

Put

$$\mathrm{P}_5 := \{(\beta, \mu); \beta \in \mathrm{End}(G_0), \mu \in \mathrm{Hom}(G_2, G_1); \phi_{2,0}\beta = \mu\phi_{1,0}\}.$$

Let us link up the set P_5 with $\mathrm{End}(S)$ by defining a map

$$\Gamma_5 : \mathrm{P}_5 \longrightarrow \mathrm{End}(S)$$

by

$$(\beta, \mu)\Gamma_5 = \underline{\beta, \mu}_5.$$

It is clear that the map Γ_5 is one-to-one. Let us define the group

$$\overline{\mathrm{P}_5} := \mathrm{gp}\langle \mathrm{P}_5 \rangle.$$

Then we can extend the map Γ_5 to $\Gamma_5^\star$ where

$$\Gamma_5^\star : \overline{\mathrm{P}_5} \longrightarrow E(S)$$

is defined by

$$(a, b)\Gamma_5^\star = \textstyle\sum_{i=1}^{n} \epsilon_i \underline{\beta_i, \mu_i}_5$$

where, $a = \sum_{i=1}^{n} \epsilon_i \beta_i$ and $b = \sum_{i=1}^{n} \epsilon_i \mu_i$, $\epsilon_i = \pm 1$.
It can be shown that the map $\Gamma_5^\star$ is a monomorphism, and consequently there is an isomorphic copy of the group $\overline{\mathrm{P}}_5$, E^{f_5} say, which is generated by the endomorphisms of type f_5, inside $E(S)$. Also we can see that E^{f_5} is a subdirect product of two groups.

Put

$$\mathrm{P}_6 := \{(\alpha, \mu); \alpha \in \mathrm{End}(G_1), \mu \in \mathrm{Hom}(G_2, G_1); \phi_{2,0}\bar{\alpha}\psi = \mu\phi_{1,0}\}.$$

Let us link up the set P_6 with $\mathrm{End}(S)$ by defining a map

$$\Gamma_6 : \mathrm{P}_6 \longrightarrow \mathrm{End}(S)$$

by

$$(\alpha, \mu)\Gamma_6 = \underline{\alpha, \mu}_6.$$

Again it is easy to see that the map Γ_6 is one-to-one. Let us define the group

$$\overline{\mathrm{P}_6} := \mathrm{gp}\langle \mathrm{P}_6 \rangle.$$

The map Γ_6 can be extended to $\Gamma_6^\star$ where

$$\Gamma_6^\star : \overline{\mathrm{P}_6} \longrightarrow E(S)$$

is defined by

$$(x,y)\Gamma_6^\star = \sum_{i=1}^n \epsilon_i \underline{\alpha_i, \mu_i}_6$$

where, $x = \sum_{i=1}^n \epsilon_i \alpha_i$ and $y = \sum_{i=1}^n \epsilon_i \mu_i, \epsilon_i = \pm 1$.

It can be shown that the map $\Gamma_6^\star$ is a monomorphism, and hence the endomorphisms of type f_6 on S will generate in $E(S)$ an isomorphic copy of the group $\overline{\mathrm{P}}_6$, which we denote by E^{f_6}. Moreover, E^{f_6} is a subdirect product of two groups.

Put

$$\mathrm{P}_7 := \{(\omega, \lambda); \omega \in \mathrm{End}(G_2), \lambda \in \mathrm{Hom}(G_1, G_2); \phi_{1,0}\hat{\omega}\chi = \lambda\phi_{2,0}\}.$$

We now link up the set P_7 with $\mathrm{End}(S)$ by defining a map

$$\Gamma_7 : \mathrm{P}_7 \longrightarrow \mathrm{End}(S)$$

by

$$(\omega, \lambda)\Gamma_7 = \underline{\omega, \lambda}_7.$$

It is clear that the map Γ_7 is one-to-one. Let us define the group

$$\overline{\mathrm{P}_7} := \mathrm{gp}{<}\mathrm{P}_7{>}.$$

Then we can extend the map Γ_7 to $\Gamma_7^\star$ such that

$$\Gamma_7^\star : \overline{\mathrm{P}_7} \longrightarrow E(S)$$

is defined by

$$(a,b)\Gamma_7^\star = \sum_{i=1}^n \epsilon_i \underline{\omega_i, \lambda_i}_7$$

where, $a = \sum_{i=1}^n \epsilon_i \omega_i$ and $b = \sum_{i=1}^n \epsilon_i \lambda_i, \epsilon_i = \pm 1$.

It can be seen that the map $\Gamma_7^\star$ is a monomorphism, and consequently there is an isomorphic copy of the group $\overline{\mathrm{P}}_7$, E^{f_7} say, which is generated by the endomorphisms of type f_7, inside $E(S)$. Also E^{f_7} is a subdirect product of two groups.

Let

$$\mathrm{End}_{K_2}(G_2) := \{\omega \in \mathrm{End}(G_2); \mathrm{K}_2\omega \subseteq \mathrm{K}_2\}$$

and let

$E_{K_2}(G_2)$ be the d.g. near-ring generated by $\mathrm{End}_{K_2}(G_2)$.

Define a map

$$\Gamma_8 : \mathrm{End}_{K_2}(G_2) \longrightarrow \mathrm{End}(S)$$

by

$$(\omega)\Gamma_8 = \underline{\omega}_8.$$

The map Γ_8 is a homomorphism, which can be extended to the map $\Gamma_8^\star$ where

$$\Gamma_8^\star : E_{K_2}(G_2) \longrightarrow E(S)$$

is defined by

$$(\sum \epsilon_i\omega_i)\Gamma_8^\star = \sum \epsilon_i \underline{\omega_{i_8}}$$

where $\epsilon_i = \pm 1$.

We can see that the map $\Gamma_8^\star$ is a monomorphism, which will assert that the endomorphisms of type f_8 generate in $E(S)$ a subnear-ring isomorphic to the near-ring $E_{K_2}(G_2)$.

Next for the endomorphisms of type f_9 we consider the set

$$\mathrm{P}_9 := \{(\beta, \lambda, \mu); \beta \in \mathrm{End}(G_0), \lambda \in \mathrm{Hom}(G_1, G_2), \mu \in \mathrm{Hom}(G_2, G_1); \\ \phi_{1,0}\beta = \lambda\phi_{2,0}, \phi_{2,0}\beta = \mu\phi_{1,0}\}.$$

Now we link up the set P_9 with $\mathrm{End}(S)$ by defining a map

$$\Gamma_9 : \mathrm{P}_9 \longrightarrow \mathrm{End}(S)$$

by

$$(\beta, \lambda, \mu)\Gamma_9 = \underline{\beta, \lambda, \mu}_9.$$

It is clear that the map Γ_9 is one-to-one. Let us define the group

$$\overline{\mathrm{P}_9} := \mathrm{gp}{<}\mathrm{P}_9{>}.$$

Then we can extend the map Γ_9 to $\Gamma_9^\star$ where

$$\Gamma_9^\star : \overline{\mathrm{P}_9} \longrightarrow E(S)$$

is defined by

$$(a, b, c)\Gamma_9^\star = \textstyle\sum_{i=1}^n \epsilon_i \underline{\beta_i, \lambda_i, \mu_{i_9}}$$

where, $a=\sum_{i=1}^{n}\epsilon_i\beta_i$, $b=\sum_{i=1}^{n}\epsilon_i\lambda_i, c=\sum_{i=1}^{n}\epsilon_i\mu_i, \epsilon_i=\pm 1$.

It can be shown that the map Γ_9^* is a monomorphism, and consequently there is an isomorphic copy, in $E(S)$, of the group $\overline{\mathrm{P}}_9$,which is generated by the endomorphisms of type f_9 and denoted by E^{f_9}. Furthermore, E^{f_9} is a subdirect product of three groups.

Now we consider the set

$$\mathrm{End}_{K_1}(G_1) := \{\alpha\in \mathrm{End}(G_1); \mathrm{K}_1\alpha\subseteq \mathrm{K}_1\ \}$$

and let

$E_{K_1}(G_1)$ be the d.g. near-ring generated by $\mathrm{End}_{K_1}(G_1)$.

Define a map

$$\Gamma_{10} : \mathrm{End}_{K_1}(G_1)\longrightarrow \mathrm{End}(S)$$

by

$$(\alpha)\Gamma_{10}=\underline{\alpha}_{10}.$$

We show that the map Γ_{10} is a homomorphism. Let $\alpha_1,\alpha_2\in \mathrm{End}_{K_1}(G_1)$, then for $s_i\in G_i$, where $i=0,1,2$, we have

$$\begin{aligned}
s_2(\alpha_1\alpha_2)\Gamma_{10} &= s_2(\alpha)\Gamma_{10}, \text{where } \alpha=\alpha_1\alpha_2, &(10.38)\\
&= s_2\underline{\alpha}_{10}\\
&= s_2\phi_{2,0}\overline{\alpha}\psi\\
&= s_2\phi_{2,0}\overline{\alpha}_1\overline{\alpha}_2\psi\\
&= s_2\phi_{2,0}(\overline{\alpha}_1\psi)(\overline{\alpha}_2\psi)\\
&= s_2\underline{\alpha_1}_{10}\overline{\alpha}_2\psi\\
&= s_2\underline{\alpha_1}_{10}\underline{\alpha_2}_{10}\\
&= s_2(\alpha_1\Gamma_{10})(\alpha_2\Gamma_{10}) &(10.39)
\end{aligned}$$

$$
\begin{aligned}
s_1(\alpha_1\alpha_2)\Gamma_{10} &= s_1(\alpha)\Gamma_{10}, \text{where } \alpha = \alpha_1\alpha_2, && (10.40)\\
&= s_1\underline{\alpha}_{10}\\
&= s_1\alpha\\
&= s_1\alpha_1\alpha_2\\
&= s_1\underline{\alpha_1}_{10}\alpha_2\\
&= s_1\underline{\alpha_1}_{10}\underline{\alpha_2}_{10}\\
&= s_1(\alpha_1\Gamma_{10})(\alpha_2\Gamma_{10}), && (10.41)
\end{aligned}
$$

$$
\begin{aligned}
s_0(\alpha_1\alpha_2)\Gamma_{10} &= s_0(\alpha)\Gamma_{10}, \text{where } \alpha = \alpha_1\alpha_2,\\
&= s_0\underline{\alpha}_{10}\\
&= s_0\overline{\alpha}\psi && (10.42)\\
&= s_0\overline{\alpha}_1\overline{\alpha}_2\psi\\
&= s_0(\overline{\alpha}_1\psi)(\overline{\alpha}_2\psi)\\
&= s_0\underline{\alpha_1}_{10}\underline{\alpha_2}_{10}\\
&= s_0(\alpha_1\Gamma_{10})(\alpha_2\Gamma_{10}). && (10.43)
\end{aligned}
$$

Equations (10.38) — (10.43) show that $(\alpha_1\alpha_2)\Gamma_{10} = (\alpha_1\Gamma_{10})(\alpha_2\Gamma_{10})$ and Γ_{10} is a homomorphism.

Next we extend the map Γ_{10} to $\Gamma_{10}^\star$ where

$$\Gamma_{10}^\star : E_{K_1}(G_1) \longrightarrow E(S)$$

is defined by

$$(\sum \epsilon_i\alpha_i)\Gamma_{10}^\star = \sum \epsilon_i\underline{\alpha_i}_{10}$$

where $\epsilon_i = \pm 1$.

The map $\Gamma_{10}^\star$ is a homomorphism. To see this let us suppose that $0 = c \in E_{K_1}(G_1)$, where $c = \sum_{i=1}^n \epsilon_i\alpha_i$. We show that $(c)\Gamma_{10}^\star = 0$. By assumption we have $g_1\sum_{i=1}^n \epsilon_i\alpha_i = 0$ for all $g_1 \in G_1$. Thus for $g_2 \in G_2$, we have

$$g_2(c)\Gamma_{10}^\star = g_2(\sum_{i=1}^n \epsilon_i\alpha_i)\Gamma_{10}^\star$$

$$
\begin{aligned}
&= g_2 \sum_{i=1}^{n} \epsilon_i \underline{\alpha}_{i_{10}} \\
&= \sum_{i=1}^{n} \epsilon_i g_2 \underline{\alpha}_{i_{10}} \\
&= \sum_{i=1}^{n} \epsilon_i (g_2 \phi_{2,0}) \bar{\alpha}_i \psi \\
&= \sum_{i=1}^{n} \epsilon_i g_0 \bar{\alpha}_i \psi, \text{ where } g_0 = g_2 \phi_{2,0}, \\
&= \sum_{i=1}^{n} \epsilon_i (g_1 \phi_{1,0}) \bar{\alpha}_i \psi, \text{ where } g_1 \phi_{1,0} = g_0, \\
&= g_1 \sum_{i=1} \epsilon_i \phi_{1,0} \bar{\alpha}_i \psi \\
&= g_1 (\sum_{i=1}^{n} \epsilon_i \alpha_i) \phi_{1,0} \\
&= 0
\end{aligned}
$$

and for $g_1 \in G_1$, we have

$$
\begin{aligned}
g_1(c)\Gamma^\star_{10} &= g_1(\sum_{i=1}^{n} \epsilon_i \alpha_i)\Gamma^\star_{10} \\
&= g_1(\sum_{i=1}^{n} \epsilon_i \underline{\alpha}_{i_{10}}) \\
&= \sum_{i=1}^{n} \epsilon_i g_1 \underline{\alpha}_{i_{10}} \\
&= \sum_{i=1}^{n} \epsilon_i g_1 \alpha_i \\
&= g_1 \sum_{i=1}^{n} \epsilon_i \alpha_i \\
&= 0
\end{aligned}
$$

and finally for $g_0 \in G_0$, we have

$$
\begin{aligned}
g_0(c)\Gamma^\star_{10} &= g_0(\sum_{i=1}^{n} \epsilon_i \alpha_i)\Gamma^\star_{10} \\
&= g_0(\sum_{i=1}^{n} \epsilon_i \underline{\alpha}_{i_{10}}) \\
&= \sum_{i=1}^{n} \epsilon_i g_0 \underline{\alpha}_{i_{10}} \\
&= \sum_{i=1}^{n} \epsilon_i (g_0 \bar{\alpha}_i \psi) \\
&= \sum_{i=1}^{n} \epsilon_i (g_1 \phi_{1,0}) \bar{\alpha}_i \psi
\end{aligned}
$$

$$\begin{aligned} &= \sum_{i=1}^{n} \epsilon_i(g_1\alpha_i\phi_{1,0}), \text{by } (10.6)\,, \\ &= g_1(\sum_{i=1}^{n} \epsilon_i\alpha_i)\phi_{1,0} \\ &= 0. \end{aligned}$$

This shows that $(c)\Gamma_{10}^{\star} = 0$ and $\Gamma_{10}^{\star}$ is a homomorphism.

Next, we show that $\Gamma_{10}^{\star}$ is a monomorphism.

Suppose that $c_1 \neq c_2$ in $E_{K_1}(G_1)$, where $c_1 = \sum_{i=1}^{n} \epsilon_i\alpha_i$ and $c_2 = \sum_{j=1}^{r} \eta_j v_j$, then there exists $g_1 \in G_1$ such that

$$\begin{aligned} g_1\sum_{i=1}^{n} \epsilon_i\alpha_i &\neq g_1\sum_{j=1}^{r} \eta_j v_j \\ \sum_{i=1}^{n} \epsilon_i g_1\alpha_i &\neq \sum_{j=1}^{r} \eta_j g_1 v_j \\ \sum_{i=1}^{n} \epsilon_i g_1\underline{\alpha_i}_{10} &\neq \sum_{j=1}^{r} \eta_j g_1\underline{v_j}_{10} \\ g_1\sum_{i=1}^{n} \epsilon_i\underline{\alpha_i}_{10} &\neq g_1\sum_{j=1}^{r} \eta_j\underline{v_j}_{10} \\ g_1(\sum_{i=1}^{n} \epsilon_i\alpha_i)\Gamma_{10}^{\star} &\neq g_1(\sum_{j=1}^{r} \eta_j v_j)\Gamma_{10}^{\star} \\ (c_1)\Gamma_1^{\star} &\neq (c_2)\Gamma_{10}^{\star}. \end{aligned}$$

Hence $\Gamma_{10}^{\star}$ is a monomorphism.

We have shown that the endomorphisms of S of type f_{10} generate in $E(S)$ a subnear-ring isomorphic to the near-ring $E_{K_1}(G_1)$.

Finally for the endomorphisms of type f_{11} we put

$$\mathrm{P}_{11} := \{(\beta, \lambda); \beta \in \mathrm{End}(G_0), \lambda \in \mathrm{Hom}(G_1, G_2); \phi_{1,0}\beta = \lambda\phi_{2,0}\}.$$

We now link up the set P_{11} with $\mathrm{End}(S)$ by defining a map

$$\Gamma_{11} : \mathrm{P}_{11} \longrightarrow \mathrm{End}(S)$$

by

$$(\beta, \lambda)\Gamma_{11} = \underline{\beta, \lambda}_{11}.$$

It is easily seen that the map Γ_{11} is one-to-one. Let us define the group

$$\overline{\mathrm{P}_{11}} := \mathrm{gp}<\mathrm{P}_{11}>.$$

Then we can extend the map Γ_{11} to $\Gamma^{\star}_{11}$ such that

$$\Gamma^{\star}_{11} : \overline{\mathrm{P}_{11}} \longrightarrow E(S)$$

is defined by

$$(a,b)\Gamma^{\star}_{11} = \underline{\textstyle\sum_{i=1}^{n} \epsilon_i\beta_i, \lambda_i}_{11}$$

where, $a = \sum_{i=1}^{n} \epsilon_i\beta_i, b = \sum_{i=1}^{n} \epsilon_i\lambda_i, \epsilon_i = \pm 1.$

It can be shown that the map $\Gamma^{\star}_{11}$ is a monomorphism, which will assert that the endomorphisms of S of type f_{11} will generate in $E(S)$ an isomorphic copy of the group $\overline{\mathrm{P}}_{11}$, which we denote by $E^{f_{11}}$. Also $E^{f_{11}}$ is a subdirect product of two groups.

Next, as might be expected, we describe addition in $E(S)$.

10.2 Addition in $E(S)$

Let us assume that $\underline{\alpha,\omega}_1$ and $\underline{\upsilon,\mu}_6$ are two endomorphisms of S of type f_1 and f_6 respectively. Then for $s_2 \in G_2$, we have

$$\begin{aligned}
s_2(\underline{\alpha,\omega}_1 + \underline{\upsilon,\mu}_6) &= s_2\underline{\alpha,\omega}_1 + s_2\underline{\upsilon,\mu}_6 \\
&= s_2\omega + s_2\mu \\
&= s_2\omega\phi_{2,0} + s_2\mu\phi_{1,0} \\
&= s_2\phi_{2,0}\bar{\alpha}\psi + s_2\mu\phi_{1,0}, \text{ by (10.4) and (10.5)}, \\
&= s_2\phi_{2,0}\bar{\alpha}\psi + s_2\phi_{2,0}\bar{\upsilon}\psi, \text{ by (10.20)}, \\
&= s_2\underline{\alpha}_{10} + s_2\underline{\upsilon}_{10} \\
&= s_2(\underline{\alpha}_{10} + \underline{\upsilon}_{10}). \qquad (10.44)
\end{aligned}$$

Also for $s_1 \in G_1$, we have

$$\begin{aligned}
s_1(\underline{\alpha,\omega}_1 + \underline{\upsilon,\mu}_6) &= s_1\underline{\alpha,\omega}_1 + s_1\underline{\upsilon,\mu}_6 \\
&= s_1\alpha + s_1\upsilon \\
&= s_1\underline{\alpha}_{10} + s_1\underline{\upsilon}_{10} \\
&= s_1(\underline{\alpha}_{10} + \underline{\upsilon}_{10}) \qquad (10.45)
\end{aligned}$$

and for $s_0 \in G_0$, we have

$$\begin{aligned} s_0(\underline{\alpha,\omega}_1 + \underline{\upsilon,\mu}_6) &= s_0\underline{\alpha,\omega}_1 + s_0\underline{\upsilon,\mu}_6 \\ &= s_0\bar{\alpha}\psi + s_0\bar{\upsilon}\psi \\ &= s_0\underline{\alpha}_{10} + s_0\underline{\upsilon}_{10} \\ &= s_0(\underline{\alpha}_{10} + \underline{\upsilon}_{10}). \end{aligned} \tag{10.46}$$

Equations (10.44) — (10.46) show that

$$\underline{\alpha,\omega}_1 + \underline{\upsilon,\mu}_6 = \underline{\alpha}_{10} + \underline{\upsilon}_{10}. \tag{10.47}$$

Similarly we can get

$$\underline{\upsilon,\mu}_6 + \underline{\alpha,\omega}_1 = \underline{\upsilon}_{10} + \underline{\alpha}_{10}. \tag{10.48}$$

This shows that the sum of two endomorphisms of S of type f_1 and type f_6 lies in $E(G)^{f_{10}}$. Following the same pattern, we can obtain the following table of addition :

$+$	f_1	f_2	f_3	f_4	f_5	f_6	f_7	f_8	f_9	f_{10}	f_{11}
f_1	1	2	10	8	2	10	8	8	2	10	2
f_2	2	2	2	2	2	2	2	2	2	2	2
f_3	10	2	3	2	5	6	2	2	5	10	2
f_4	8	2	2	4	2	2	7	8	11	2	11
f_5	2	2	5	2	5	5	2	2	5	2	2
f_6	10	2	6	2	5	6	2	2	5	10	2
f_7	8	2	2	7	2	2	7	8	11	2	11
f_8	8	2	2	8	2	2	8	8	2	2	2
f_9	2	2	5	11	5	5	11	2	9	2	11
f_{10}	10	2	10	2	2	10	2	2	2	10	2
f_{11}	2	2	2	11	2	2	11	2	11	2	11

The above sum of different types of endomorphisms can be generalized so that we get a precise description of addition in $E(S)$. This can be done using induction arguments similar to those used in case V in chapter 9 and we get the same table of addition in $E(S)$ which was obtained on page 124. Now we turn to describe the product in $E(S)$.

10.3 Product in $E(S)$

Let us consider the endomorphisms $\underline{\beta,\mu}_5$ and $\underline{\alpha}_{10}$ as two endomorphisms of S of type f_5 and type f_{10} respectively. Recall that equation (10.19) is satisfied for $\underline{\beta,\mu}_5$ and equation (10.6) is satisfied for $\underline{\alpha}_{10}$, then we have

$$\begin{aligned} \phi_{2,0}\beta\bar{\alpha}\psi &= \mu\phi_{1,0}\bar{\alpha}\psi, \text{by (10.19)}, \\ &= \mu\alpha\phi_{1,0}, \text{by (10.6)}. \end{aligned} \tag{10.49}$$

Let $g_2 \in G_2$, then

$$\begin{aligned} g_2\underline{\beta,\mu}_5\underline{\alpha}_{10} &= (g_2\mu)\underline{\alpha}_{10}. \\ &= g_2\mu\alpha. \end{aligned}$$

Equation (10.49) enables us to write the right hand side of the above equation as

$$g_2\underline{\beta,\mu}_5\underline{\alpha}_{10} = g_2\underline{\beta\bar{\alpha}\psi,\mu\alpha}_5. \tag{10.50}$$

Now let $g_1 \in G_1$, then

$$\begin{aligned} g_1\underline{\beta,\mu}_5\underline{\alpha}_{10} &= (g_1\phi_{1,0}\beta)\underline{\alpha}_{10} \\ &= g_1\phi_{1,0}\beta\bar{\alpha}\psi \\ &= g_1\underline{\beta\bar{\alpha}\psi,\mu\alpha}_5, \text{since (10.49) is satisfied.} \end{aligned} \tag{10.51}$$

Now let $g_0 \in G_0$, then

$$\begin{aligned} g_0\underline{\beta,\mu}_5\underline{\alpha}_{10} &= (g_0\beta)\underline{\alpha}_{10} \\ &= g_0\beta\bar{\alpha}\psi \\ &= g_1\underline{\beta\bar{\alpha}\psi,\mu\alpha}_5, \text{since (10.49) is satisfied.} \end{aligned} \tag{10.52}$$

Equations (10.50) — (10.52) imply that

$$\underline{\beta,\mu}_5\underline{\alpha}_{10} = \underline{\beta\bar{\alpha}\psi,\mu\alpha}_5. \tag{10.53}$$

Now we reverse the order of the maps in the above product to get for $g_2 \in G_2$

$$\begin{aligned} g_2\underline{\alpha}_{10}\underline{\beta,\mu_5} &= (g_2\phi_{2,0}\bar{\alpha}\psi)\underline{\beta,\mu_5} \\ &= g_2\phi_{2,0}\bar{\alpha}\psi\beta \\ &= g_2\underline{\bar{\alpha}\psi\beta_2} \end{aligned} \tag{10.54}$$

and for $g_1 \in G_1$, we have

$$\begin{aligned} g_1\underline{\alpha}_{10}\underline{\beta,\mu_5} &= (g_1\alpha)\underline{\beta,\mu_5} \\ &= g_1\alpha\phi_{1,0}\beta \\ &= g_1\phi_{1,0}\bar{\alpha}\psi\beta, \text{by (10.6)}, \\ &= g_2\underline{\bar{\alpha}\psi\beta_2} \end{aligned} \tag{10.55}$$

and for $g_0 \in G_0$, we have

$$\begin{aligned} g_0\underline{\alpha}_{10}\underline{\beta,\mu_5} &= g_0(\bar{\alpha}\psi)\underline{\beta,\mu_5} \\ &= g_0\bar{\alpha}\psi\beta \\ &= g_0\underline{\bar{\alpha}\psi\beta_2}. \end{aligned} \tag{10.56}$$

Equations (10.54) — (10.56) imply that

$$\underline{\alpha}_{10}\underline{\beta,\mu_5} = \underline{\bar{\alpha}\psi\beta_2}. \tag{10.57}$$

Equation (10.53) shows that the product $\underline{\beta,\mu_5}\underline{\alpha}_{10}$ lies in E^{f_5} while equation (10.57) shows that the product $\underline{\alpha}_{10}\underline{\beta,\mu_5}$ lies in E^{f_2}. This means that the product fails to yield a semilattice of semigroups. In fact, $E(S)$ has the same table of product which was given on page 126.

10.4 Conclusion VI

Let us return to the sum in $E(S)$. Let

$$\mathcal{L} = \{f_q\,;\, q \in \{1, 2, \ldots, 11\}\}.$$

Recall the endomorphism f_u which was defined in (8.6), so that we get a similar situation to (9.17). For each pair E^{f_i}, E^{f_j} such that $f_i \geq f_j$, we define the linking homomorphisms ϕ_{f_i,f_j} where

$$\phi_{f_i,f_j} : E^{f_i} \longrightarrow E^{f_j}$$

is given by

$$\textstyle\sum_r \epsilon_r \underline{\alpha_{r_i}} \longrightarrow \sum_r \epsilon_r \underline{\alpha_{r_j}} \ .$$

Then we have a strong semilattice of groups given by

$$E(S) = (\mathcal{L}, \{E^{f_q}\}_{f_q \in \mathcal{L}}, \{\phi_{f_i,f_j}\})$$

that is,

$(E(S), +)$ is a Clifford semigroup.

This semilattice can be described by the same figure which is on page 128.

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